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Continuous Optimization

The Karush–Kuhn–Tucker optimality conditions in multiobjective programming problems with interval-valued objective functions

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ABSTRACT

The KKT conditions in multiobjective programming problems with interval-valued objective functions are derived in this paper. Many concepts of Pareto optimal solutions are proposed by considering two orderings on the class of all closed intervals. In order to consider the differentiation of an interval-valued function, we invoke the Hausdorff metric to define the distance between two closed intervals and the Hukuhara difference to define the difference of two closed intervals. Under these settings, we are able to consider the continuity and differentiability of an interval-valued function. The KKT optimality conditions can then be naturally elicited.

1. Introduction

Imposing the uncertainty upon the optimization problems is an interesting research topic. The uncertainty may be interpreted as randomness or fuzziness. The randomness occurring in the optimization problems is categorized as the stochastic optimization problems. The books written by Birge and Louveaux [5], Kall [18], Prékopa [26], Stancu-Minasian [34] and Vajda [38] provide many interesting ideas and useful techniques for tackling the stochastic optimization problems. On the other hand, the fuzziness occurring in the optimization problems is categorized as the fuzzy optimization problems. The collection of papers on fuzzy optimization edited by Słowiński [29] and Delgado et al. [10] gives the main stream of this topic. Lai and Hwang [19,20] also give an insightful survey. The fusion of randomness and fuzziness occurring in the optimization problems is even a challenge research topic. The book edited by Słowiński and Teghem [30] gives the comparisons between fuzzy optimization and stochastic optimization for the multiobjective programming problems. Inuiguchi and Ramík [15] also gives a brief review of fuzzy optimization and a comparison with stochastic optimization in portfolio selection problem.

As we have known in the stochastic optimization problems, the coefficients of the problem are assumed as random variables with known distributions in most of cases. Also, in the fuzzy optimization problems, the coefficients of the problem are frequently assumed as fuzzy numbers with known membership functions. However, the specifications of the distributions and membership functions in the stochastic optimization problems and fuzzy optimization problems, respectively, are very subjective. For example, many researchers invoke the Gaussian (normal) distributions with different parameters in the stochastic optimization problems, and the bell-shaped or S-shaped membership functions in the fuzzy optimization problems. These specifications may not perfectly match the real situations. Therefore, interval-valued optimization problems may provide an alternative choice for considering the uncertainty into the optimization problems. That is to say, the coefficients in the interval-valued optimization problems are assumed as closed intervals. Although the specifications of closed intervals may still be judged as subjective viewpoint, we might argue that the bounds of uncertain data (i.e., determining the closed intervals to bound the possible observed data) are easier to be handled than specifying the distributions and membership functions in stochastic optimization and fuzzy optimization problems, respectively.

We consider a simple example to motivate our study. Suppose that a factory can produce five products $x_1, \ldots, x_5$ subject to some budget constraints. According to the past experience and the expert's assessment, for selling product x_i, $i = 1, \ldots, 5$, the factory can earn income that will be between a_i^L and a_i^U, where $a_i^L < a_i^U$ for $i = 1, \ldots, 5$. However, for producing product x_i, the factory will consume some resources. According to the workers' experience, producing product x_i, $i = 1, \ldots, 5$, will consume the quantity of resources that will be between b_i^L and

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b_i^U, where $b_i^L < b_i^U$ for $i = 1, \ldots, 5$. The decision-makers of this factory will try to maximize the income and minimize the consumption of resources subject to some budget constraints. In this case, the problem can be formulated as follows:

$$\begin{aligned} &\min && (f_1(x_1,\ldots,x_5), f_2(x_1,\ldots,x_5)) \\ &\text{subject to} && g_i(x_1,\ldots,x_5) \leqslant c_i, \quad i = 1,\ldots,m, \\ &&& x_1,x_2,x_3,x_4,x_5 \geqslant 0, \end{aligned}$$

where each c_i is the upper limit of the budget of event i, and f_1 and f_2 are interval-valued functions given by

$$f_1(x_1,\ldots,x_5) = -[a_1^L, a_1^U]x_1 \ominus [a_2^L, a_2^U]x_2 \ominus [a_3^L, a_3^U]x_3 \ominus [a_4^L, a_4^U]x_4 \ominus [a_5^L, a_5^U]x_5$$

and

$$f_2(x_1,\ldots,x_5) = [b_1^L, b_1^U]x_1 \oplus [b_2^L, b_2^U]x_2 \oplus [b_3^L, b_3^U]x_3 \oplus [b_4^L, b_4^U]x_4 \oplus [b_5^L, b_5^U]x_5,$$

where $\oplus$ and $\ominus$ means the addition and subtraction of two closed intervals, respectively.

The duality theory for inexact linear programming problem was proposed by Soyster [31–33] and Thuente [36]. Falk [11] provided some properties on this problem. However, Pomerol [25] pointed out some drawbacks of Soyster's results and also provided some mild conditions to improve Soyster's results.

The interval-valued optimization problems are closely related with the inexact linear programming problems. Charnes et al. [8] considered the linear programming problems in which the right-hand sides of linear inequality constraints were taken as closed intervals. Steuer [35] proposed three algorithms, called the F-cone algorithm, E-cone algorithm and all emanating algorithm to solve the linear programming problems with interval objective functions. Bitran [6] discussed the connectedness of the set of efficient extreme points and the existence of efficient points in the multiobjective linear programming problems with interval coefficients, where an implicit enumeration algorithm was also proposed to obtain the solutions. Ishibuchi and Tanaka [17] proposed the ordering relation between two closed intervals by considering the maximization and minimization problems separately. Inuiguchi and Kume [14] formulated and solved four kinds of goal programming problems with interval coefficients in which the target values were also assumed as closed intervals. Inuiguchi and Sakawa [16] introduced the concept of minimax regret solution and investigated its relation to the possibly and necessarily optimal solutions. Mráz [23] provided algorithms to compute the exact upper and lower bounds of optimal solutions for linear programming problems with interval coefficients. Chinneck and Ramadan [9] proposed an approach to find the best optimum and worst optimum, and the point settings of the interval coefficients that can achieve these two extremes. Sengupta et al. [28] proposed the $\mathscr{A}$-index to compare two closed intervals and used it to define a satisfactory crisp equivalent system of an inequality constraint with interval coefficients, where the concept of satisfactory solution was also defined. Urli and Nadeau [37] used an interactive method STEM to solve the multiobjective linear programming problems with interval coefficients, where they also proposed a methodology to transform a nondeterministic problems into the deterministic problems. Chanas and Kuchta [7] presented an approach to unify the solution methods proposed by Ishibuchi and Tanaka [17] and Rommelfanger et al. [27]. Also the portfolio selection problem with interval objective functions were investigated by Ida [13]. Recently, Oliveria and Antunes [24] provided an overview of multiobjective linear programming problems with interval coefficients by illustrating many numerical examples.

In this paper, we provide many solution concepts for the multiobjective programming problem with interval-valued objective functions. One of the solution concepts follows from Ishibuchi and Tanaka [17]. Two ordering relationships between two closed intervals in $\mathbb{R}$ are proposed. We shall see that these two orderings are not total orderings on the class of all closed intervals. Therefore, we shall invoke the solution concept (Pareto optimal solution) in the (conventional) multiobjective programming problems to deal with the multiobjective programming problems with interval-valued objective functions. Under these settings, we are going to derive the Karush–Kuhn–Tucker optimality condition. The KKT optimality conditions for the optimization problem (single-objective programming problem) with interval-valued objective function was investigated by Wu [39]. However, in this paper, more solution concepts and interesting techniques are proposed and presented than those of Wu [39].

In Section 2, we introduce some basic properties and arithmetics of intervals. We invoke the well-known Hausdorff metric to define the distance between any two closed intervals. Using this metric, we can consider the continuity and limit of an interval-valued function. Then we invoke the Hukuhara difference to define the difference of any two closed intervals. Using this Hukuhara difference and the concept of limit in interval-valued function, we are capable of proposing the differentiation of an interval-valued function. In Section 3, we formulate the multiobjective programming problems with interval-valued objective functions and provide many solution concepts for this problem. In Section 4, we derive the KKT conditions based on the solution concepts proposed in Section 3. In Section 5, a numerical example is illustrated for providing the basic techniques to compute the Pareto optimal solutions by resorting to KKT conditions.

2. Preliminaries

Let $\mathscr{K}_c(\mathbb{R})$ denote the class of all nonempty, compact and convex subsets of $\mathbb{R}$. Let $A, B \in \mathscr{K}_c(\mathbb{R})$. Then $A + B$ is defined by $A + B = \{a + b : a \in A \text{ and } b \in B\}$ and $-A$ is defined by $-A = \{-a : a \in A\}$. Therefore, $A - B = A + (-B)$.

Let us denote by $\mathscr{I}$ the class of all closed intervals in $\mathbb{R}$. If A is a closed interval, we also adopt the notation $A = [a^L, a^U]$, where a^L and a^U means the lower and upper bounds of A, respectively. Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be in $\mathscr{I}$. Then, by definition, we have

(i) $A + B = \{a + b : a \in A \text{ and } b \in B\} = [a^L + b^L, a^U + b^U]$;
(ii) $-A = \{-a : a \in A\} = [-a^U, -a^L]$.

Therefore, we see that $A - B = A + (-B) = [a^L - b^U, a^U - b^L]$. We also see that

$$kA = \{ka : a \in A\} = \begin{cases} [ka^L, ka^U] & \text{if } k \geqslant 0, \\ [ka^U, ka^L] & \text{if } k < 0, \end{cases}$$

where k is a real number. For more details on the topic of interval analysis, we refer to Moore [21,22] and Alefeld and Herzberger [1].