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Discrete Optimization

Parametric mixed-integer 0–1 linear programming: The general case for a single parameter

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Abstract

Two algorithms for the general case of parametric mixed-integer linear programs (MILPs) are proposed. Parametric MILPs are considered in which a single parameter can simultaneously influence the objective function, the right-hand side and the matrix. The first algorithm is based on branch-and-bound on the integer variables, solving a parametric linear program (LP) at each node. The second algorithm is based on the optimality range of a qualitatively invariant solution, decomposing the parametric optimization problem into a series of regular MILPs, parametric LPs and regular mixed-integer nonlinear programs (MINLPs). The number of subproblems required for a particular instance is equal to the number of critical regions. For the parametric LPs an improvement of the well-known rational simplex algorithm is presented, that requires less consecutive operations on rational functions. Also, an alternative based on predictor–corrector continuation is proposed. Numerical results for a test set are discussed.

Keywords: Parametric programming; Post-optimality sensitivity analysis; Matrix case; MILP; MINLP

1. Introduction

Mathematical programs often involve unknown parameters and the task of parametric optimization is, in principle, to solve the mathematical program for each possible value of these unknown parameters. Discretization of the parameter range is not rigorous in general, since there is no guarantee for optimality between the mesh points. Moreover, discretization on a fine mesh is a very expensive procedure, especially for high dimension parameter spaces. Algorithms for parametric optimization typically divide the parameter range into regions of optimality, also called areas [1], or critical regions [2]. For one parameter the boundary between critical regions is called a breakpoint. For each critical region either the problem is infeasible or a qualitatively invariant solution, typically a smooth function of the parameters, is optimal. The notion of qualitatively invariant solution depends on the specific case. In parametric mixed-integer linear programs (MILPs), the topic of this paper, it means an optimal integer realization along with an optimal basis for the linear program (LP) resulting with this integer realization.

Parametric optimization has several applications [2] including waste management [3], fleet planning [4], model-predictive control [5] and process synthesis under uncertainty [6–8]. Recently, Balas and Saxena [9] used good feasible solutions to parametric MILPs to generate cuts for MILPs. Also recently, Eppstein [10] introduced the notion of inverse parametric optimization where the values of parameters that result in a given solution are searched for. Wallace [11] has argued that

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parametric optimization is valuable for decision making only when the value of the parameters is not known during the optimization phase but known during the decision making phase.

In this paper the general case of parametric MILPs is considered

$$
\begin{aligned}
f^*(p) = \min_{\mathbf{x},\mathbf{y}} & (\mathbf{c}^x(p))^T\mathbf{x} + (\mathbf{c}^y(p))^T\mathbf{y} \\
\text{s.t. } & \mathbf{A}^{1x}(p)\mathbf{x} + \mathbf{A}^{1y}(p)\mathbf{y} = \mathbf{b}^1(p), \\
& \mathbf{A}^{2x}(p)\mathbf{x} + \mathbf{A}^{2y}(p)\mathbf{y} \leqslant \mathbf{b}^2(p), \\
& \mathbf{x}^{\mathrm{L}} \leqslant \mathbf{x} \leqslant \mathbf{x}^{\mathrm{U}}, \\
& \mathbf{x} \in \mathbb{R}^{n_x}, \mathbf{y} \in \{0,1\}^{n_y},
\end{aligned} \tag{1}
$$

where $\mathbf{c}^x(p) \in \mathbb{R}^{n_x}$, $\mathbf{c}^y(p) \in \mathbb{R}^{n_y}$, $\mathbf{A}^{1x}(p) \in \mathbb{R}^{m_1 \times n_x}$, $\mathbf{A}^{1y}(p) \in \mathbb{R}^{m_1 \times n_y}$, $\mathbf{A}^{2x}(p) \in \mathbb{R}^{m_2 \times n_x}$, $\mathbf{A}^{2y}(p) \in \mathbb{R}^{m_2 \times n_y}$, $\mathbf{b}^1(p) \in \mathbb{R}^{m_1}$, $\mathbf{b}^2(p) \in \mathbb{R}^{m_2}$. Note that in deviation from the standard form finite upper and nonzero lower bounds on the variables as well as inequality constraints are allowed, to show how these can be treated efficiently. Note that in some cases the discussion is restricted to finite bounds $\mathbf{x}^{\mathrm{L}}, \mathbf{x}^{\mathrm{U}} \in \mathbb{R}^{n_x}$, while in other cases infinite bounds are allowed. As is done in most algorithmic contributions, the host set of the parameter P is assumed to be the interval $[0,1]$. This is essentially equivalent to the assumption of a compact host set, excluding unbounded parameter ranges.

Two interesting special cases of (1), that are often considered in the literature, are the cost-vector case and the right-hand side case. In both of these cases the matrices $\mathbf{A}^{1x}, \mathbf{A}^{2x}, \mathbf{A}^{1y}, \mathbf{A}^{2y}$ do not depend on the parameter p; in the former case also the right-hand side vectors $\mathbf{b}^1, \mathbf{b}^2$ are parameter independent, while in the latter case the cost-vectors $\mathbf{c}^x, \mathbf{c}^y$ are parameter independent. The cost-vector case has the benign property that the feasibility region does not depend on the parameter. As a consequence the optimality regions of a given basis are (convex) polyhedra and the optimal solution function is piecewise-affine and concave [12]. In the right-hand side case the optimality region of a given basis can be calculated relatively easily [13,14].

When the integer variables in (1) are fixed (to 0 or 1), or relaxed (to $[0,1]$), a parametric LP is obtained, which is an important problem in its own right, and can also be used as a subproblem for the solution of (1).

Parametric optimization is a mature field. Most of the theoretical properties were established by the 1980's and in recent years the focus has been on algorithmic contributions, which is also the focus of this paper. There are several textbooks and review articles on parametric optimization; Dinkelbach [15] and Gal [2] consider LP; Bank et al. [16] and Fiacco [17] consider the nonlinear case; Geoffrion and Nauss [18] discuss MILP; Greenberg [19] provides a bibliography of contributions to parametric mixed-integer programs. Algorithms for the right-hand side case with an affine dependence on one or many parameters exist for MILPs [3,7,13,18,20–22] and also for (mixed-integer) nonlinear programs [23–25]. For the cost-vector case of MILPs with an affine dependence on a single parameter a well-known algorithm is based on intersections of the objective functions of feasible points [3,26,27]. Note also that in principle (1) can be reformulated to a right-hand side problem [23,25] by introducing an auxiliary variable z and the constraint $z = p$; however in general this leads to a nonconvex parametric MINLP.

Based on the possible number of optimality regions, Murty [28] shows that the complexity of parametric optimization cannot be bounded above by a polynomial even in the right-hand side case of parametric LPs with a single parameter. Therefore, rather than basing the computational complexity on the size of the instance, it is probably more appropriate to compare the computational requirement of an algorithm with the computational requirement of solving as many regular optimization problems (at fixed parameter values) as there are optimality regions. For instance, in the cost-vector case of MILP with a single parameter the intersection-based algorithm [3,26,27] requires a number of MILP calls that is less than twice the number of optimality regions of the particular instance.

The algorithmic approaches for parametric MILP can be divided into two broad classes. In the first class, algorithms for the solution of a regular MILP are altered to solve the parametric MILP. For instance Ohtake and Nishida [20] solve the right-hand side case of parametric MILP by a branch-and-bound (B&B) on the integer variables with a parametric LP at each node. Methods based on this principle have the promise of being relatively computationally efficient if the formulated parametric subproblems are only slightly more expensive than their regular counterparts. The other broad class is to use MILP calls for fixed parameter values and process the result post-optimaly. This is, for instance, employed in the well-known intersection-based algorithm for the cost-vector case. Methods based on this principle can take advantage of state-of-the-art MILP solvers and are also relatively easy to implement.

To our best knowledge no algorithm exists for the solution of the general case of parametric MILPs, apart from our conference presentations [29,30], and extension of the available algorithms for the right-hand side and cost-vector case is nontrivial because the general case does not have the benign properties of these special cases. The most relevant contributions are on parametric LPs. Post-optimal sensitivity analysis of the matrix coefficients of nonbasic columns is covered in linear programming textbooks, e.g., [31]. Freund [32] proposes to obtain post-optimal sensitivity information through Taylor series expansions and Greenberg [33] considers post-optimal sensitivity analysis from interior solutions via duality. Gal [2] reviews the case that a single column or a single row of the matrix depends on the parameter; in this case an analytical inversion of the parametric matrix is possible based on a formula by Bodewig [34]. Dinkelbach [15] proposes an