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Invited Review

Random assignment problems

Pavlo A. Krokhmal [a,*], Panos M. Pardalos [b]

[a] Department of Mechanical and Industrial Engineering, The University of Iowa, 3131 Seamans Center, Iowa City, IA 52242, USA
[b] Department of Industrial and Systems Engineering, University of Florida, 303 Weil Hall, Gainesville, FL 32611, USA

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Abstract

Analysis of random instances of optimization problems provides valuable insights into the behavior and properties of problem's solutions, feasible region, and optimal values, especially in large-scale cases. A class of problems that have been studied extensively in the literature using the methods of probabilistic analysis is represented by the assignment problems, and many important problems in operations research and computer science can be formulated as assignment problems. This paper presents an overview of the recent results and developments in the area of probabilistic assignment problems, including the linear and multidimensional assignment problems, quadratic assignment problem, etc.

Keywords: Random assignment problems; Linear assignment problem; Quadratic assignment problem; Multidimensional assignment problem; Bottleneck assignment problem; Probabilistic analysis; Asymptotic analysis; Fitness landscape analysis

1. Introduction

Analysis of random instances of optimization problems is instrumental for elucidating the properties of problem's solutions, feasible region, and optimal values, especially in large-scale cases. The probabilistic framework, where it is assumed that the problem's data is drawn from some probability distribution, provides a facility for maintaining a consistent structure among problem instances of different sizes, thus enabling one to analyze the problem's optimal value, solutions, etc., as functions of the problem size. A particular class of combinatorial optimization problems that have been studied extensively in the probabilistic context is represented by the assignment, or matching problems.

In an assignment problem, one is looking to find an assignment, or matching, between the elements of two (or more) sets, such that the total cost of all matched pairs (tuples) is minimized. Depending on the particular structure of the sets being matched, the form of the cost function, the matching rule, and so on, the assignment problems are categorized into linear, quadratic, bottleneck, multidimensional, etc. The assignment problems can be stated in a variety of forms, including mathematical programming, combinatorial, or graph-theoretic formulations, and constitute one of the most important and fundamental objects in computer science, operations research, and discrete mathematics. Besides these areas of their "natural habitat," assignment problems have found numerous applications in other disciplines of science and engineering, including chemistry, biology, physics, archeology, electrical engineering, sports, and others (for a comprehensive review of the subject of assignment problems, their formulations and applications, see, for instance, Pardalos and Pitsoulis, 2000; Burkard, 2002; Pentico, 2007, and references therein).

The studies of random instances of assignment problems date back to as early as (Donath, 1969), where limiting behavior of the linear assignment problem was investigated by solving small (by today's standards) randomly generated instances. In retrospect, probabilistic analysis of the linear assignment problem, whose deterministic instances

* Corresponding author. Tel.: +1 319 335 5680.
E-mail addresses: krokhmal@engineering.uiowa.edu (P.A. Krokhmal), pardalos@ufl.edu (P.M. Pardalos).

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are considered as being computationally "easy", has turned out to be quite challenging to researchers. Until recently, much more has been known in the literature about the behavior of many random combinatorial optimization problems, including assignment problems, that are regarded as "difficult" and belong to the NP-hard class (e.g., the quadratic assignment problem, traveling salesman problem, etc.) than about the random instances of the "easy" linear assignment problem. This situation has changed only in the last few years, when a number of powerful results have been obtained in the scope of the random linear assignment problem, including results for *finite-size* instances, which juxtapose the predominantly *asymptotic* analyses contained in the literature on random optimization problems.

It also must be noted that studies of random assignment problems have led to important findings in the context of other optimization problems (and vice versa). The striking insight of Burkard and Fincke (1982a) that random large-scale instances of the quadratic assignment problem, which is considered one of the hardest combinatorial optimization problems, can be solved almost to optimality by practically any heuristic algorithm with high probability, has led to discovery of an entire class of combinatorial optimization problems with similar properties (Burkard and Fincke, 1985; Szpankowski, 1995). Similarly, Karp's (1987) derivation of an upper bound on the expected optimal value of the linear assignment problem was generalized to a broad class of random linear programming problems (Dyer et al., 1986), etc.

In this survey, we present a detailed exposition of the state-of-the art results in the area of random assignment problems, including the recent developments in the context of the random linear assignment problem (Section 2) and its higher-dimensional generalization, the multidimensional assignment problem (Section 3), the quadratic assignment problem (Section 4), and the bottleneck assignment problem (Section 5). Finally, Section 6 contains results on assignment problems that do not fall into the broad classes mentioned above, but for which probabilistic analysis has been conducted in the literature.

Our main focus is on the analytical results pertinent to various aspects of the corresponding random assignment problems: the optimal value, its convergence and limits, properties of the problem's landscape, local extrema, etc. The computational and algorithmic sides of the probabilistic analysis of assignment problems are covered only briefly, as a number of comprehensive surveys on solution methods for various classes of assignment problems, which also discuss probabilistic-based algorithms, are available in the literature (see, among others, Burkard and Çela, 1999; Burkard, 2002; Anstreicher, 2003; Loiola et al., 2007).

As many authors point out (see, e.g., Rhee, 1988; Albrecher et al., 2006), the methods employed by different researchers to tackle the random optimization problems are often problem-specific and may be quite intricate and involved as well.[1] This, as well as the diversity of the mathematical tools employed by different authors in their studies, makes it impractical to present detailed derivations of all the findings discussed here. Instead, we tried to summarize shortly one's approach, where appropriate, while maintaining the rigorousness in the formulations of the corresponding results.

2. Linear assignment problem

The linear assignment problem (LAP)[2] is one of the basic and fundamental models in operations research, computer science, and discrete mathematics. In its most familiar interpretation, it answers the question of finding an assignment of n workers to n jobs that has the lowest total cost, if the cost of assigning worker i to task j equals c_{ij}. Apart from the straightforward applications, such as personnel assignment problems, the LAP frequently arises as a part of other optimization problems, such as quadratic assignment problem, multidimensional assignment problem, traveling salesman problem, etc. Other applications of the LAP, including earth–satellite systems with TDMA protocol, and tracking objects in space are considered in Burkard (1985) and Brogan (1989); for a more comprehensive discussion of the applications of the LAP refer to, e.g., Burkard and Çela (1999).

The mathematical programming formulation of the LAP has the form:

$$\begin{aligned} L_n = \min \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}, \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_{ij} = 1, \quad j = 1,\ldots,n, \\ & \sum_{j=1}^{n} x_{ij} = 1, \quad i = 1,\ldots,n, \end{aligned} \tag{1a}$$

where the decision variables x_{ij} can be taken as either binary: $x_{ij} \in \{0,1\}$, or non-negative: $x_{ij} \geqslant 0$, leading to an integer programming (IP) or linear programming (LP) formulations of the LAP, respectively. In the graph-theoretical setting, the LAP corresponds to finding a minimum-cost perfect matching in an edge-weighed bipartite graph; another useful interpretation of the LAP presents it as finding such a permutation of rows and columns of the cost matrix $C = (c_{ij})$ that minimizes the sum of the elements on the diagonal. The latter observation leads to the permutation formulation of the LAP:

[1] As W. Rhee acknowledges, one of her papers (Rhee, 1988) has been partially motivated by an attempt to classify the arguments employed by Frenk et al. (1985) in their study of the convergence of the optimal value of random quadratic assignment problem.

[2] Many authors use a more precise term *linear sum assignment problem* (LSAP) to distinguish it from other problems where the total cost of assignment is linear, but not equal to the sum of the costs of individual assignments (matchings): e.g., linear bottleneck assignment problem, etc.