Available online at www.sciencedirect.com
ScienceDirect
European Journal of Operational Research 186 (2008) 288–299
EUROPEAN JOURNAL OF OPERATIONAL RESEARCH
www.elsevier.com/locate/ejor

Decision Support

Shapley mappings and the cumulative value for n-person games with fuzzy coalitions

Dan Butnariu [a], Tomáš Kroupa [b,*]

[a] Department of Mathematics, University of Haifa, 31 905 Haifa, Israel
[b] Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Pod Vodárenskou věží 4, 182 08 Prague, Czech Republic

Received 28 July 2006; accepted 28 January 2007
Available online 20 February 2007

Abstract

In this paper we prove existence and uniqueness of the so-called Shapley mapping, which is a solution concept for a class of n-person games with fuzzy coalitions whose elements are defined by the specific structure of their characteristic functions. The Shapley mapping, when it exists, associates to each fuzzy coalition in the game an allocation of the coalitional worth satisfying the efficiency, the symmetry, and the null-player conditions. It determines a "cumulative value" that is the "sum" of all coalitional allocations for whose computation we provide an explicit formula.

Keywords: Coalition; Cumulative value; Fuzzy coalition; n-Person cooperative game; Shapley value

1. Introduction

Let $N := \{1, 2, \ldots, n\}$ be the set whose elements are called *players*. As usual, by a *coalition* we mean a subset of N. A *fuzzy coalition* is a vector $A = (A(1), \ldots, A(n))$ with coordinates $A(i)$ contained in the interval $[0, 1]$ (cf. [1,2]). The number $A(i)$ is called the *membership degree of player i* to the fuzzy coalition A. We denote by $\mathscr{P}$ the set of all coalitions and by $\mathscr{F}$ the set of all fuzzy coalitions. When referring to coalitions we do not notationally distinguish between a coalition S and its indicator vector $(S(1), \ldots, S(n))$, where the coordinates $S(i)$ are either one or zero depending on whether i belongs or not to S. In this way we can view $\mathscr{P}$ as a subset of $\mathscr{F}$. A fuzzy coalition A can be also seen as a partition of the set of players into coalitions

$$A_t := \{i \in N : A(i) = t\}, \quad t \in [0, 1],$$

such that all players belonging to A_t for some $t \in [0, 1]$ have the same degree of membership to A. Clearly, all but at most n coalitions A_t are non-empty.

* Corresponding author. Tel.: +420 266 052 595.
E-mail address: kroupa@utia.cas.cz (T. Kroupa).

0377-2217/$ - see front matter
doi:10.1016/j.ejor.2007.01.033

A (*characteristic function n-person cooperative*) *game* is a function $u : \mathscr{P} \to \mathbb{R}$ such that $u(\emptyset) = 0$. The function u associates to each coalition S its *worth* $u(S)$, measuring the utility of forming coalition S. We presume that, besides coalitions, formation of fuzzy coalitions in the game u is also possible: the worth of a fuzzy coalition A is the aggregated worth of the coalitions A_t weighted by a quantity $\psi(t)$ which depends on the degree of membership t. In other words, the worth of a fuzzy coalition A in the game u is given by

$$u^{\psi}(A) = \sum_{t\in[0,1]} \psi(t)u(A_t). \tag{1}$$

Note that the sum occurring here is well-defined since all but finitely many terms of it are zero. In this context it is natural to assume that the function $\psi : [0,1] \to \mathbb{R}$ is such that the coalition A_1 of fully fledged members of A gets its full worth while the coalition A_0 of players who are not members of A does not contribute to the worth $u^{\psi}(A)$. Therefore, all over this paper we make the following assumption:

Assumption 1. $(\psi(t) = 0 \iff t = 0)$ and $(\psi(1) = 1)$.

A function $\psi : [0,1] \to \mathbb{R}$ with this property is called a *weight function*.

In what follows, a function $v : \mathscr{F} \to \mathbb{R}$ satisfying $v(\emptyset) = 0$ is called a *fuzzy game* – cf. [1,2]. We denote by $\mathscr{G}[\psi]$ the set of fuzzy games v satisfying

$$v(A) = \sum_{t\in[0,1]} \psi(t)v(A_t). \tag{2}$$

It is easy to see that, if $v \in \mathscr{G}[\psi]$, then the restriction u of v to $\mathscr{P}$ is a game such that $v = u^{\psi}$. The game u is called *the underlying game of* v. Clearly, a fuzzy game $v \in \mathscr{G}[\psi]$ and its underlying game u completely determine each other. Also, observe that $\mathscr{G}[\psi]$ is a linear space with the usual operations induced from $\mathbb{R}$. It is worth mentioning that the class of games $\mathscr{G}[\psi]$ represents a certain scheme for calculating a profit of a fuzzy coalition, which is justified and developed in detail from the economic point of view in Example 1 of Section 4.

A first question we are dealing with in this paper is whether, in games in which formation of fuzzy coalitions is possible and the worth of each fuzzy coalition is determined according to (2), there are ways of ‘‘fairly'' distributing the worth of all fuzzy coalitions among the players. Of course, the answer to this question essentially depends on the meaning of ‘‘fairness''. In order to make this precise, recall (cf. [9,10]) that if v is a fuzzy game and if A is a fuzzy coalition, then the fuzzy coalition B is called a *v-carrier of A* if the following two conditions are satisfied:

(i) $B_t \subseteq A_t$ for every $t \in (0,1]$;
(ii) if $C \in \mathscr{F}$ and $C_t \subseteq A_t$ for every $t \in (0,1]$, then $v(B_t \cap C_t) = v(C_t)$ for every $t \in (0,1]$.

As usual, for every permutation π of N, every $A \in \mathscr{F}$, and any fuzzy game v, we denote $\pi A := A \circ \pi^{-1}$ and $\pi v(A) := v(\pi^{-1}A)$. Clearly, if v belongs to $\mathscr{G}[\psi]$, then the function $\pi v : A \mapsto \pi v(A)$ from $\mathscr{F}$ to $\mathbb{R}$ is still a fuzzy game in $\mathscr{G}[\psi]$. With these in mind we can define the following notion which describes a concept of fairness according to which each fuzzy coalition allocates its worth to its members obeying the principles intrinsic to the Shapley value, that is, efficiency, null-players get nothing, symmetry, and linearity (see [13]).

Definition 1. A *Shapley mapping* is a linear function $\mathbf{\Phi} : \mathscr{G}[\psi] \to (\mathbb{R}^N)^{\mathscr{F}}$ satisfying the following conditions for any $v \in \mathscr{G}[\psi]$ and any $A \in \mathscr{F}$:

Axiom 1. (*Coalitional Efficiency*) For every v-carrier $B \in \mathscr{F}$ of A we have

$$\sum_{i\in N:B(i)>0} \mathbf{\Phi}_i(v)(A) = v(B). \tag{3}$$

Axiom 2 (*Non-Member*). If $A(j) = 0$, then $\mathbf{\Phi}_j(v)(A) = 0$.

Axiom 3 (*Symmetry*). If π is a permutation of N, then