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Discrete Optimization

On combinatorial optimization problems on matroids with uncertain weights

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Abstract

In this paper the combinatorial optimization problem on weighted matroid is considered. It is assumed that the weights in the problem are ill-known and they are modeled as fuzzy intervals. The optimality of solutions and the optimality of elements are characterized. This characterization is performed in the setting of possibility theory. A method of choosing a solution under uncertainty is also proposed.

Keywords: Matroid; Interval; Fuzzy interval; Possibility theory; Combinatorial optimization

1. Introduction

In the combinatorial optimization problem we seek an object composed of the elements of a given finite set E. In a typical situation a deterministic, real weight is associated with every element of the set E and we seek an object for which the total weight is minimal or maximal. Many problems of both practical and theoretical importance can be formulated in this way. A standard example is the calculation of a minimum spanning tree or a shortest path in a given graph, where E is the set of edges of this graph. Over the past few decades a collection of techniques of solving such problems has emerged. For a comprehensive review we refer the reader to [1,20].

An important and interesting class of combinatorial optimization problems can be formulated on matroids. The name matroid was introduced by Whitney in 1935 [22]. Since then it has been recognized that matroids arise naturally in combinatorial optimization. Matroids are precisely the structures for which the very simple and efficient greedy algorithm works. Perhaps, the best known example is the minimum (maximum) spanning

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tree problem, for which the greedy algorithm is known in the literature as Kruskal's algorithm [1]. A good introduction to matroids can be found in [19].

In the classical problem, it is assumed that all the weights are precisely known. However, this assumption may be a serious restriction since in many practical applications the exact values of the weights are not known in advance. This uncertainty is caused by the varying nature of the world. A typical example is traveling time between two cities in the shortest path problem or activity durations in scheduling problems.

One of the simplest methods of modeling the imprecise weights is to define them as closed intervals. It is then assumed that the value of each weight may fall within a given range, independently on the values taken by the other weights. The classical interval can be generalized to the *fuzzy interval*, which is richer in information. The evaluation by fuzzy intervals is performed in the setting of possibility theory. The possibility theory, which proposes a natural framework for handling incomplete knowledge, is fully described in [10].

The first problem connected with imprecise weights is the characterization of the optimality of solutions and the optimality of elements. If the exact values of the weights are not available, then the notion "optimal solution" also becomes imprecise. It is not possible to say a priori which solution will be optimal and which elements of E will be included in the optimal solution. However, using possibility theory, we can calculate the possibility and the necessity of an event that a given solution will be optimal or a given element will be a part of an optimal solution. This approach is not new. It was first applied to analyze the criticality of paths and activities in fuzzy CPM (see [6,12]), thus it was applied to the longest path problem in a directed and acyclic network. In [5] the optimality of sequences of jobs in the single machine scheduling problem with fuzzy parameters was characterized. Finally, in [23] the optimality of trees and edges in the minimum spanning tree problem with interval weights was studied. The authors in [23] do not use directly possibility theory (nor fuzzy intervals as well) but their results have a clear possibilistic interpretation. The minimum spanning tree problem is an example of a matroidal problem and one of the main aims of this paper is to generalize the results obtained in [23] for all matroidal problems.

The second problem connected with imprecise weights is choosing a solution. Since the weights are uncertain, it is clear that an additional criterion is required to perform this task. If the weights are modeled as classical intervals, then one of the most natural criteria is the *maximal regret*, which expresses the maximal deviation of a given solution from the optimum. The maximal regret criterion is described in book [17], which is entirely devoted to robust discrete optimization. It has been applied to some matroidal problems in [2,3,8,23]. Unfortunately, most of the combinatorial optimization problems with the maximal regret criterion (in particular the minimum spanning tree problem) turned out to be strongly $\mathcal{NP}$-hard. Moreover, the minmax regret approach is valid only if the parameters are modeled as the classical intervals. In this paper we study the concept of the necessarily optimal solution, which is closely related to the maximal regret and it can be also applied to fuzzy intervals. We show how to calculate a solution for which the necessity of being optimal is maximal. This solution can be naturally chosen as the solution under uncertainty. It can be viewed as the most stable and the most resistant to deteriorations. This approach is closely related to stability analysis [21]. In stability analysis the situation in which several parameters of the problem can vary simultaneously is considered. For a given solution a stability radius is calculated which indicates how a given solution is resistant to deteriorations. In [21] some results on the minimum spanning tree problem are presented.

This paper is organized as follows. In Section 2, we recall the definition of a matroid and we formulate the classical combinatorial optimization problem on a weighted matroid. In Section 3, we consider the case in which the weights are given by means of the classical intervals. We formulate several problems and we construct efficient algorithms for all of them. In Section 4, we show how to generalize the problems defined in Section 3 to the fuzzy case. We also provide a precise interpretation of the fuzzy interval.

2. Preliminaries

Consider a system $(E, \mathcal{I})$, where $E = \{e_1, \ldots, e_n\}$ is a nonempty ground set and $\mathcal{I}$ is a collection of subsets of E closed under inclusion, that is:

(I1) if $B \in \mathcal{I}$ and $A \subseteq B$ then $A \in \mathcal{I}$.