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Decision Support

A note on "A goal programming model for incomplete interval multiplicative preference relations and its application in group decision-making"

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ABSTRACT

In a recently published paper by Liu et al. [Liu, F., Zhang, W.G., Wang, Z.X. (2012). A goal programming model for incomplete interval multiplicative preference relations and its application in group decision-making. European Journal of Operational Research 218, 747–754], two equations are introduced to define consistency of incomplete interval multiplicative preference relations (IMPRs) and employed to develop a goal programming model for estimating missing values. This note illustrates that such consistency definition and estimation model are technically incorrect. New transitivity conditions are proposed to define consistent IMPRs, and a two-stage goal programming approach is devised to estimate missing values for incomplete IMPRs.

1. Introduction

The interval multiplicative preference relation (IMPR) introduced by Saaty and Vargas (1987) is a powerful and efficient tool for expressing decision-makers' (DMs') pairwise judgments with uncertainty, and has been used to model uncertainty in multi-criteria decision analysis (Durbach & Stewart, 2012). On the other hand, because of complexity of decision problems and the limited ability of human pairwise comparisons, DMs may be unable to provide their preferences over some pairs of decision alternatives. As such, incomplete judgments are resulted and decision making problems with incomplete preference relations have been received more and more attention in recent years (Mattila & Virtanen, 2015; Punkka & Salo, 2013; Wang & Li, 2015). The consistency of preference relations is a crucial foundation for estimating missing values and obtaining a reasonable and reliable decision result (Brunelli, Canal, & Fedrizzi, 2013; Brunelli & Fedrizzi, 2015; Kou, Ergu, & Shang, 2014). In order to characterize inconsistency indices for pairwise comparison matrices, Brunelli and Fedrizzi (2014) put forward five axioms. One of them is the invariance with respect to permutations of decision alternatives.

In a recent paper, Liu et al. (2012) introduced two equations to define consistency of incomplete IMPRs, and developed a goal programming model to estimate missing values. However, a close investigation demonstrates that such consistency definition is highly dependent on alternative labels and not robust to permutations of decision alternatives, and the goal programming model suffers from serious drawbacks. The aim of this note is to point out and correct errors in the consistency definition and the goal programming model.

At first, the terminology and notation used are mainly introduced as follows.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of n alternatives, an IMPR $\bar{A}$ on X is characterized by an interval judgment matrix $\bar{A} = (\bar{a}_{ij})_{n\times n} = ([a_{ij}^-, a_{ij}^+])_{n\times n}$ with

$$1/S \le a_{ij}^- \le a_{ij}^+ \le S,\ a_{ij}^- a_{ji}^+ = 1,\ a_{ii}^- = a_{ii}^+ = 1, \quad i, j = 1, 2, \ldots, n \tag{1.1}$$

where $\bar{a}_{ij}$ is an interval judgment selected in a bounded scale $[1/S, S]$, and indicates that x_i is between a_{ij}^- and a_{ij}^+ times preferred than x_j.

The pairwise judgments are usually expressed on the 1-9 Saaty's scale, i.e. $S = 9$. If some judgments in $\bar{A}$ are missing, then $\bar{A}$ is referred to as an incomplete IMPR. Obviously, the missing values may be lower or/and upper bounds of interval judgments.

2. The invalidity of the consistency definition and the goal programming model by Liu et al. (2012)

Liu et al. (2012) employed two formulae (see Liu et al. (2012), Eq. (6) on page 748) to construct two multiplicative preference

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relations $C = (c_{ij})_{n\times n}$ and $D = (d_{ij})_{n\times n}$ from an IMPR $\bar{A} = (\bar{a}_{ij})_{n\times n} = ([a_{ij}^-, a_{ij}^+])_{n\times n}$. The formulae can be rewritten as follows by using the notation in this note

$$c_{ij} = \begin{cases} a_{ij}^+ & i<j \\ 1 & i=j \\ a_{ij}^- & i>j \end{cases} \quad d_{ij} = \begin{cases} a_{ij}^- & i<j \\ 1 & i=j \\ a_{ij}^+ & i>j \end{cases} \tag{2.1}$$

When $\bar{A}$ is an incomplete IMPR, C and D constructed from $\bar{A}$ by using (2.1) may be two incomplete comparison matrices. Based on (2.1), Liu et al. (2012) proposed the consistency definition for incomplete IMPRs (see Liu et al. (2012), Definition 7 on page 749) as follows:

Definition 2.1. (Liu et al., 2012). Let $\bar{A} = (\bar{a}_{ij})_{n\times n} = ([a_{ij}^-, a_{ij}^+])_{n\times n}$ be an incomplete IMPR. $\bar{A}$ is called consistent, if all the known elements of C and D defined by (2.1) satisfy the following condition:

$$c_{ij} = c_{ik}c_{kj},\ d_{ij} = d_{ik}d_{kj} \qquad i,j,k = 1,2,\ldots,n \tag{2.2}$$

Next, an example is provided to illustrate that such consistency definition is technically wrong.

Example 1. Consider a decision making problem with four alternatives. Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of the alternatives. After comparing pairs of the alternatives, a DM furnishes an incomplete IMPR as follows:

$$\bar{A}_1 = (\bar{a}_{ij})_{4\times 4} = \left([a_{ij}^-, a_{ij}^+]\right)_{4\times 4}$$
$$= \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} [1,1] & [2,3] & [3,6] & [1,3] \\ [1/3,1/2] & [1,1] & [3/2,2] & - \\ [1/6,1/3] & [1/2,2/3] & [1,1] & [1/3,1/2] \\ [1/3,1] & - & [2,3] & [1,1] \end{bmatrix} \end{array}$$

where "–" indicates the missing or unknown values.

As per (2.1), the two incomplete multiplicative preference relations are determined as

$$C_1 = (c_{ij})_{4\times 4} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 3 & 6 & 3 \\ 1/3 & 1 & 2 & - \\ 1/6 & 1/2 & 1 & 1/2 \\ 1/3 & - & 2 & 1 \end{bmatrix} \end{array}$$

$$D_1 = (d_{ij})_{4\times 4} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1/2 & 1 & 3/2 & - \\ 1/3 & 2/3 & 1 & 1/3 \\ 1 & - & 3 & 1 \end{bmatrix} \end{array}$$

One can easily verify that C_1 and D_1 satisfy (2.2). According to Definition 2.1, $\bar{A}_1$ is a consistent and incomplete IMPR.

Let σ be a permutation of $\{1, 2, 3, 4\}$ satisfying $\sigma(1) = 1, \sigma(2) = 4, \sigma(3) = 3, \sigma(4) = 2$, then we have the following incomplete IMPR.

$$\bar{A}'_1 = (\bar{a}'_{ij})_{4\times 4} = \left([a'^-_{ij}, a'^+_{ij}]\right)_{4\times 4} = \left([a^-_{\sigma(i)\sigma(j)}, a^+_{\sigma(i)\sigma(j)}]\right)_{4\times 4}$$
$$= \begin{array}{c} \\ x_1 \\ x_4 \\ x_3 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_4 & x_3 & x_2 \end{array} \\ \begin{bmatrix} [1,1] & [1,3] & [3,6] & [2,3] \\ [1/3,1] & [1,1] & [2,3] & - \\ [1/6,1/3] & [1/3,1/2] & [1,1] & [1/2,2/3] \\ [1/3,1/2] & - & [3/2,2] & [1,1] \end{bmatrix} \end{array}$$

Similarly, by (2.1), the two incomplete multiplicative preference relations C'_1 and D'_1 are obtained as:

$$C'_1 = \left(c'_{ij}\right)_{4\times 4} = \begin{array}{c} \\ x_1 \\ x_4 \\ x_3 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_4 & x_3 & x_2 \end{array} \\ \begin{bmatrix} 1 & 3 & 6 & 3 \\ 1/3 & 1 & 3 & - \\ 1/6 & 1/3 & 1 & 2/3 \\ 1/3 & - & 3/2 & 1 \end{bmatrix} \end{array}$$

$$D'_1 = \left(d'_{ij}\right)_{4\times 4} = \begin{array}{c} \\ x_1 \\ x_4 \\ x_3 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_4 & x_3 & x_2 \end{array} \\ \begin{bmatrix} 1 & 1 & 3 & 2 \\ 1 & 1 & 2 & - \\ 1/3 & 1/2 & 1 & 1/2 \\ 1/2 & - & 2 & 1 \end{bmatrix} \end{array}$$

As $c'_{13} = 6 \neq 9 = 3 \times 3 = c'_{12}c'_{23}$ and $d'_{13} = 3 \neq 2 = 1 \times 2 = d'_{12}d'_{23}$, C'_1 and D'_1 do not satisfy (2.2). By Definition 2.1, $\bar{A}'_1$ is an inconsistent and incomplete IMPR.

Example 1 clearly indicates that, for the identical judgment information with different labeling for alternatives, Definition 2.1 yields contradictory results. In other words, Definition 2.1 is not robust to permutations of decision alternatives. From the viewpoint of the pairwise comparison, the consistency of preference relations should be independent of alternative labels such that it has the invariance with respect to permutations of decision alternatives. Therefore, Definition 2.1 by Liu et al. (2012) is technically incorrect.

In addition, incomplete IMPRs with extremely inconsistent judgments may be determined as consistent IMPRs as per Definition 2.1. Moreover, incorrect results may be obtained when the matrices C and D defined by (2.1) are used to check acceptability and estimate missing values for incomplete IMPRs.

Example 2. Consider the following two incomplete IMPRs:

$$\bar{A}_2 = \left([a_{ij}^-, a_{ij}^+]\right)_{3\times 3} = \begin{bmatrix} [1,1] & [9,9] & [-,1/7] \\ [1/9,1/9] & [1,1] & [8,-] \\ [7,-] & [-,1/8] & [1,1] \end{bmatrix}$$

$$\bar{A}_3 = \left([a_{ij}^-, a_{ij}^+]\right)_{3\times 3} = \begin{bmatrix} [1,1] & [2,2] & [-,1/4] \\ [1/2,1/2] & [1,1] & [3,-] \\ [4,-] & [-,1/3] & [1,1] \end{bmatrix}$$

For $\bar{A}_2$, the given judgment $a_{12} = [9, 9]$ indicates that x_1 is 9 times preferred than x_2, and the interval judgment $a_{23} = [8, -]$ denotes that x_2 is at least 8 times preferred than x_3, whereas, the interval judgment $a_{31} = [7, -]$ gives that x_3 is at least 7 times preferred than x_1. Therefore, the judgments in $\bar{A}_2$ are extremely inconsistent. However, by Definition 2.1, one can easily obtain that $\bar{A}_2$ is a consistent IMPR. Obviously, this result is highly questionable.

For $\bar{A}_3$, as per (2.1), the two incomplete comparison matrices are determined as

$$C_3 = \begin{bmatrix} 1 & 2 & 1/4 \\ 1/2 & 1 & - \\ 4 & - & 1 \end{bmatrix} \qquad D_3 = \begin{bmatrix} 1 & 2 & - \\ 1/2 & 1 & 3 \\ - & 1/3 & 1 \end{bmatrix}$$

As per Definition 2.1, the incomplete IMPR $\bar{A}_3$ is consistent. By using (2.2), the obtained estimation values are $c_{23} = 1/8$ and $d_{13} = 6$, i.e., $a_{23}^+ = 1/8$ and $a_{13}^- = 6$. However, by observing the lower bound of the interval judgment $\bar{a}_{23} = [3, -]$ in the given IMPR $\bar{A}_3$, any possible value a_{23}^+ should be more than or equal to 3. Similarly, a_{13}^- should be less than or equal to 1/4. Clearly, the results obtained by (2.2) are incorrect.

As per the concept of acceptable IMPRs by Liu et al. (2012) (see Definition 8 on page 749), one can obtain that $\bar{A}_2$ is unacceptable and $\bar{A}_3$ is acceptable, implying that a consistent IMPR may be unacceptable and an acceptably incomplete IMPR may not be complemented.