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Continuous Optimization

Algorithms for the continuous nonlinear resource allocation problem—New implementations and numerical studies

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ABSTRACT

Patriksson (2008) provided a then up-to-date survey on the continuous, separable, differentiable and convex resource allocation problem with a single resource constraint. Since the publication of that paper the interest in the problem has grown: several new applications have arisen where the problem at hand constitutes a subproblem, and several new algorithms have been developed for its efficient solution. This paper therefore serves three purposes. First, it provides an up-to-date extension of the survey of the literature of the field, complementing the survey in Patriksson (2008) with more then 20 books and articles. Second, it contributes improvements of some of these algorithms, in particular with an improvement of the pegging (that is, variable fixing) process in the relaxation algorithm, and an improved means to evaluate subsolutions. Third, it numerically evaluates several relaxation (primal) and breakpoint (dual) algorithms, incorporating a variety of pegging strategies, as well as a quasi-Newton method. Our conclusion is that our modification of the relaxation algorithm performs the best. At least for problem sizes up to 30 million variables the practical time complexity for the breakpoint and relaxation algorithms is linear.

1. Introduction

We consider the continuous, separable, differentiable and convex resource allocation problem with a single resource constraint. The problem is formulated as follows: Let $J := \{1, 2, \dots, n\}$. Let $\phi_j : \mathbb{R} \to \mathbb{R}$ and $g_j : \mathbb{R} \to \mathbb{R}, j \in J$, be convex and continuously differentiable. Moreover, let $b \in \mathbb{R}$ and $-\infty < l_j < u_j < \infty, j \in J$. Consider the problem to

$$\underset{x}{\text{minimize}} \quad \phi(\mathbf{x}) := \sum_{j \in J} \phi_j(x_j), \tag{1a}$$

$$\text{subject to} \quad g(\mathbf{x}) := \sum_{j \in J} g_j(x_j) \le b, \tag{1b}$$

$$l_j \le x_j \le u_j, \quad j \in J. \tag{1c}$$

We also consider the problem where the inequality constraint (1b) is replaced by an equality, i.e.,

$$\underset{x}{\text{minimize}} \quad \phi(\mathbf{x}) := \sum_{j \in J} \phi_j(x_j), \tag{2a}$$

$$\text{subject to} \quad g(\mathbf{x}) := \sum_{j \in J} a_j x_j = b, \tag{2b}$$

$$l_j \le x_j \le u_j, \quad j \in J, \tag{2c}$$

where $a_j \neq 0, j \in J$, and the sign is the same for all $j \in J$. Further, we assume that there exists an optimal solution to problems (1) and (2). For brevity, in the following discussions we define $X_j := [l_j, u_j], j \in J$.

Problems (1) and (2) arise in many areas, e.g., in search theory (Koopman, 1999), economics (Markowitz, 1952), stratified sampling (Bretthauer, Ross, & Shetty, 1999), inventory systems (Maloney & Klein, 1993), and queuing manufacturing networks (Bitran & Tirupati, 1989). Further, these problems occur as subproblems in algorithms that solve the integer resource allocation problem (Mjelde, 1983, Section 4.7; Ibaraki & Katoh, 1988, pp. 72–75; Bretthauer & Shetty, 2002b), multicommodity network flows (Shor, 1985, Section 4.2), and several others. Moreover, problems (1) and (2) can be used as subproblems when solving resource allocation problems with more than one resource constraint (Federgruen & Zipkin, 1983; Mjelde, 1983), and to solve extensions of problems (1) and (2) to a nonseparable objective function ϕ (Dahiya, Suneja, & Verma, 2007; Mjelde, 1983). The books Mjelde (1983), Ibaraki and Katoh (1988), and Luss (2012) describe several extensions, such as to minmax/maxmin objectives, multiple time periods, substitutable resources, network constraints, and integer decision variables.

Many numerical studies of the problems (1) and (2) have been performed; for example, see Bitran and Hax (1981), Nielsen and Zenios (1992), Robinson, Jiang, and Lemke (1992), Kodialam and Luss (1998),

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Kiwiel (2007), Kiwiel (2008a), and Kiwiel (2008b). Our numerical study is however timely and well motivated, since except for those by Kiwiel (2007, 2008a, 2008b), where the quadratic knapsack problem is studied, none of the earlier approaches study large-scale versions of problem (1) or (2). There are also several algorithms (e.g., Nielsen & Zenios, 1992, Section 1.4; Stefanov, 2001) which are claimed to be promising, but have not been evaluated in a large-scale study. Only one earlier study (Kodialam & Luss, 1998) evaluates the performance of algorithms for the problems (1) and (2) with respect to variations in the portions of the variables whose values are at a lower or upper bound at the optimal solution (see Section 6.2), and this is done for modest size instances ($n = 10^4$) only. Further, no study has been done on the computational complexity for non-quadratic versions of the problems (1) or (2). Our numerical study also incorporates improvements of the relaxation algorithm, as presented in Sections 4.3.4 and 4.3.5, and utilizes performance profiles (Dolan & Moré, 2002).

As a final note on the compuational tests, we only consider problem instances where the dual variable corresponding to the resource constraint (1b), respectively (2b), can be found in closed form; otherwise, we would need to implement a numerical method in some of the steps, e.g., a Newton method. We consider only customized algorithms for the problem at hand, since we presume that they perform better than more general algorithms under the above assumption.

Patriksson (2008) presents a survey of the history and applications of problems (1) and (2). Since its publication several related articles have been published; the survey of Patriksson (2008) is therefore complemented in Section 2. Section 3 presents a framework of breakpoint algorithms, resulting in three concrete representatives. Section 4 presents a framework of relaxation algorithms, and ultimately six concrete example methods. In Section 5 we describe a quasi-Newton method, due to Nielsen and Zenios (1992), for solving the problem (2). Section 6 describes the numerical study. A test problem set is specified and the performance profile used for the evaluation is defined. In Section 7, we analyze the results from the numerical study. The structure is such that we first compare the relaxation algorithms, second the pegging process, and third the best performing algorithms among these two with the quasi-Newton method. Finally, we draw overall conclusions.

2. Extension of the survey in Patriksson (2008)

We here extend the survey in Patriksson (2008), using the same taxonomy, and sorted according to publication date.

Mjelde (1983) K. M. Mjelde, *Methods of the allocation of limited resources*, Section 4.7

(Problem) $\phi_j \in C^2$; linear equality ($a_j = 1$); $l_j = 0$.
(Methodology) The ranking algorithm of Luss and Gupta (1975).
(Citations) Applications in capital budgeting (Hansmann, 1968; Shih, 1977), cost-effectiveness problems (Kirsch, 1968; Mjelde, 1978; Pack, 1970), health care (Fetter, 1973), marketing (Luss & Gupta, 1975), multiobjective optimization (Geoffrion, 1967), portfolio selection (Jucker & de Faro, 1975), production (the internal report leading to Bitran & Hax, 1979), reliability (Bodin, 1969), route-planning for ships or aircraft (Dantzig, Blattner, & Rao, 1966), search (Charnes & Cooper, 1958), ship loading (Kydland, 1969), and weapons selection (Danskin, 1967).
(Notes) A monograph on resource allocation problems containing a comprehensive overview of the resource allocation problem, including extensions to several resources, non-convex or non-differentiable objectives, integral decision variables, fractional programming formulations, etc.

Shor (1985) N. Z. Shor, *Minimization Methods for Nondifferentiable Functions*, Section 4.2

(Problem) $\phi_j(x_j) = \frac{1}{2}(x_j - y_j)^2$; linear equality ($a_j = 1$); $l_j = 0$.
(Methodology) Pegging.
(Citations) Shor and Ivanova (1969), in which the motivating linear programming application is described.
(Notes) The problem arises within the framework of a right-hand side allocation algorithm for a large-scale linear program.

Hua and Zhang (2005) Z.-S. Hua and B. Zhang, *Direct algorithm for separable continuous convex quadratic knapsack problem* (in Chinese)

(Problem) $\phi_j(x_j) = \frac{q_j}{2}x_j^2 - r_jx_j$; linear inequality ($a_j > 0$); $l_j = 0$.
(Methodology) Pegging.
(Citations) Algorithms for the problem (Bretthauer & Shetty, 2002a, 2002b; Melman & Rabinowitz, 2000; Pardalos & Kovoor, 1990) as well as for the case of integer variables.
(Notes) A numerical illustration ($n = 6$).

Dai and Fletcher (2006) Y.-H. Dai and R. Fletcher, *New algorithms for singly linearly constrained quadratic programs subject to lower and upper bounds.*

(Problem) $\phi_j(x_j) = \frac{q_j}{2}x_j^2 - r_jx_j$, $q_j > 0$; g_j convex in C^2 with $g'(x_j) > 0$.
(Methodology) A combination of a bracketing algorithm on the Lagrangian dual derivative, and a secant algorithm for the Lagrangian dual problem.
(Citations) Algorithms for the problem (Brucker, 1984; Calamai & Moré, 1987; Helgason, Kennington, & Lall, 1980; Pardalos & Kovoor, 1990).
(Notes) The problem arises as a subproblem in a gradient projection method for a general quadratic programming problem over a scaled simplex.

Li and Sun (2006) D. Li and X. Sun, *Nonlinear integer programming*, Chapter 6: Nonlinear Knapsack Problems, Section 6.1: Continuous–Relaxation-based Branch–and–Bound Methods.

(Problem) ϕ_j and g_j increasing; g_j convex in C^2 with $g_j' > 0$.
(Methodology) Multiplier search.
(Citations) Multiplier search methods (Bretthauer & Shetty, 1995), pegging methods (Bretthauer & Shetty, 2002a, 2002b).
(Notes) The problem arises as a subproblem in branch–and–bound methods for the integer programming version of the problem, such as for the quadratic knapsack problem, stratified sampling, manufacturing capacity planning, linearly constrained redundancy optimization in reliability networks, and linear cost minimization in reliability networks.

Dahiya et al. (2007) K. Dahiya, S. K. Suneja and V. Verma, *Convex programming with a single separable constraint and bounded variables.*

(Problem) $\phi_j(x_j) = \frac{q_j}{2}x_j^2 - r_jx_j, q_j > 0$; g_j convex in C^2 with $g'(x_j) > 0$; studies also the special case of a linear equality.
(Methodology) Iterative descent process using strictly convex quadratic separable approximations of a nonseparable original objective $f \in C^2$; subproblems solved using the pegging algorithm of Stefanov (2001).
(Citations) General references on convex programming over box constraints; Helgason et al. (1980), Dussault, Ferland, and Lemaire (1986), and Pardalos and Kovoor (1990) for example algorithms for separable convex programming.