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Continuous Optimization

A practicable branch and bound algorithm for sum of linear ratios problem

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ABSTRACT

This article presents a practicable algorithm for globally solving sum of linear ratios problem (SLR). The algorithm works by globally solving a bilinear programming problem (EQ) that is equivalent to the problem (SLR). In the algorithm, by utilizing convex envelope and concave envelope of bilinear function, the initial nonconvex programming problem is reduced to a sequence of linear relaxation programming problems. In order to improve the computational efficiency of the algorithm, a new accelerating technique is introduced, which provides a theoretical possibility to delete a large part of the investigated region in which there exists no global optimal solution of the (EQ). By combining this innovative technique with branch and bound operations, a global optimization algorithm is designed for solving the problem (SLR). Finally, numerical experimental results show the feasibility and efficiency of the proposed algorithm.

1. Introduction

The sum of linear ratios problems have broad applications in management science, system engineering, optimization designing, transportation planning, finance and investment, bond portfolio optimization, cluster analysis, engineering optimization, geometric application, computer vision, biodiversity conservation, data envelopment analysis, and so on, see Almogy and Levin (1970), Bajalinov (2003), Cploantoni, Manes, and Whinston (1969), Schaible (1996), Konno and Watanabe (1996), Drezner, Schaible, and Simchi-Levi (1990), Schaible (1981), Stancu-Minasian (1997), Rao (1971), Schaible and Shi (2003), Billionnet (2013), Du, Cook, Liang, and Zhu (2014), Jeyakumar, Li, and Srisatkunarajah (2013), Kao (2014), Lim and Zhu (2013) and Yang, Li, Chen, and Liang (2014), and whose mathematical modelling can be stated as follows:

$$(\mathrm{SLR}):\begin{cases}\max\ f(x)=\sum\limits_{i=1}^{p}\delta_i\dfrac{\varphi_i(x)}{\psi_i(x)}\\ \text{s. t. }\ x\in D\triangleq\{x\in R^n\,|\,Ax\le b,x\ge 0\},\end{cases}$$

where D is a nonempty bounded set, the numerator $\varphi_i(x)$ and the denominator $\psi_i(x)$ are all affine functions defined on R^n with assumption that $\psi_i(x)\neq 0$ for any $x\in D$, $\delta_i, i=1,2,\dots,p$, are all arbitrary real numbers. Since $\psi_i(x)$ is a continuous function defined on D, from the intermediate value theory, this implies $\psi_i(x)<0$ or $\psi_i(x)>0$ for all $x\in D$. If $\psi_i(x)<0$, we can replace $\frac{\varphi_i(x)}{\psi_i(x)}$ with $\frac{-\varphi_i(x)}{-\psi_i(x)}$, the problem remains essentially unchanged, so that we can always assume that $\psi_i(x)>0$ for all $x\in D$. Moreover, as the set of solutions is bounded, in fact, if there exists some $x\in D$ such that $\varphi_i(x)<0$, we can replace $\frac{\varphi_i(x)}{\psi_i(x)}$ with $\frac{[\varphi_i(x)+M\psi_i(x)]}{\psi_i(x)}$, where M is a constant so large that $[\varphi_i(x)+M\psi_i(x)]\ge 0$ for any $x\in D$, the problem still remains essentially unchanged. Therefore, we can also assume that $\varphi_i(x)\ge 0$. In all, without loss of generality, in the following, we can always assume that $\varphi_i(x)\ge 0$ and $\psi_i(x)>0$, $i=1,2,\dots,p$, for all $x\in D$. Naturally that some numerical problems can arise as a denominator vanishes, i.e., if there exists some ratio $\frac{\varphi_i(x)}{\psi_i(x)}$, which degrades into an affine function $\varphi_i(x)$, for this case, the affine function $\varphi_i(x)$ can be regarded as a ratio whose denominator is a constant 1, that is to say, $\varphi_i(x)$ can be looked as $\frac{\varphi_i(x)}{1}$, the problem can still be solved by using the algorithm proposed in this paper.

In last decades, various solution approaches have been proposed for globally solving the special form of the sum of linear ratios problem (SLR). For instance, parametric simplex method (Konno, Yajima, & Matsui, 1991), outer approximation method (Benson, 2010), image space method (Falk & Palocsay, 1994), unified monotonic approach (Phuong & Tuy, 2003), interior point algorithm (Nesterov & Nemirovskii, 1995), heuristic method (Konno & Abe, 1999), simplicial branch and bound duality-bounds algorithm (Benson, 2007), concave minimization method (Benson, 2004a), branch and cut technique (Costa, 2010), branch and bound algorithms (Bazaraa, Sherali, & Shetty, 2006; Jiao, 2009; Jiao & Chen, 2008; Konno & Fukaishi, 2000; Kuno, 2002, 2005; Lin, 2008; Shen & Wang, 2008; Shi, 2011; Wang & Shen, 2008; Wang, Shen, & Liang, 2005), and so on. Although many

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algorithms can be used to solve special form of sum of linear ratios problem, as far as we know, only Shen and Wang (2006) present a branch and bound algorithm for maximizing the sum of linear ratios with coefficients. In addition, several algorithms (Benson, 2002a, 2002b; Dai, Shi, & Wang, 2005; Fang, Gao, Sheu, & Xing, 2009; Gao, Mishra, & Shi, 2012; Jaberipour & Khorram, 2010; Jiao, Wang, & Chen, 2013; Pei & Zhu, 2013; Shen & Jin, 2010; Shen, Duan, & Pei, 2009; Wang & Zhang, 2004) have been developed for globally solving sum of nonlinear ratios problems.

In this paper, we will present a branch and bound algorithm for globally solving the sum of linear ratios problem (SLR) by utilizing new accelerating technique. The main features of our algorithm are given as follows. (1) We consider general sum of linear ratios problem, the investigated mathematical modeling in this article is more general than other one considered. (2) The algorithm works by globally solving a bilinear programming problem (EQ) that is equivalent to the problem (SLR). By utilizing convex envelope and concave envelope of bilinear function, the equivalent problem (EQ) is reduced to a sequence of linear relaxation programming problems. (3) In order to improve the computational efficiency of the proposed algorithm, a new accelerating technique based on outer space is raised, it offers a possibility to cut away a large part of the investigated region where there does not exist the global optimal solution of the problem (EQ), and which can be looked as an accelerating installation for global optimization of the problem (SLR). (4) The proposed algorithm economizes the required computations by conducting the branch-and-bound search in R^p rather than in R^n or R^{2p}, where p is the number of ratios in the objective function of the problem (SLR) and n is the number of decision variables in the (SLR). (5) The proposed algorithm is convergent to the global optimal solution through the successive refinement partition of the outer space region and solving a sequence of linear relaxation programming problems (LRP). Finally, numerical results show that our algorithm can be used to effectively solve the sum of linear ratios problem (SLR).

The organization of this paper is as follows. The equivalent problem (EQ) of the (SLR) and its linear relaxation programming (LRP) which is established by utilizing convex envelope and concave envelope of bilinear function are given in Section 2. Section 3 presents and validates an innovative algorithm by combining branch and bound operations with accelerating technique. In Section 4, several test examples in recent literatures and several randomly generated problem of various dimensions are used to verify the feasibility and efficiency of our algorithm, and numerical results are given. Finally, some concluding remarks are given in Section 5.

2. Linear relaxation programming

In order to globally solve the sum of linear ratios problem (SLR), first we transform the problem (SLR) into an equivalent nonconvex programs problem (EQ). In the following, our main computation is to globally solve the equivalent problem (EQ).

2.1. Equivalent problem

Without loss of generality, we assume that

$$\delta_i > 0,\ i = 1, 2, \ldots, T;\ \delta_i < 0,\ i = T+1, T+2, \ldots, p;$$
$$l_i^0 = \min_{x\in D} \varphi_i(x),\quad u_i^0 = \max_{x\in D} \varphi_i(x),\ i = 1, 2, \ldots, p;$$
$$L_i^0 = \frac{1}{\max_{x\in D} \psi_i(x)},\quad U_i^0 = \frac{1}{\min_{x\in D} \psi_i(x)},\ i = 1, 2, \ldots, p.$$

Since $\varphi_i(x)$ and $\psi_i(x)$ are all affine functions over the set D, the value of l_i^0, u_i^0, L_i^0 and U_i^0 can be easily obtained by solving linear programming problems. Obviously, for each $i = 1, 2, \ldots, p$, we have $0 \le l_i^0 \le u_i^0$, $0 < L_i^0 \le U_i^0$.

Define

$$\Omega^0 = \{(t, s) \in R^{2p} \mid l_i^0 \le t_i \le u_i^0, L_i^0 \le s_i \le U_i^0,\ i = 1, 2, \ldots, p\},$$

and consider the following equivalent nonconvex programs problem:

$$(\mathrm{EQ}): \begin{cases} \max\ g(t, s) = \sum_{i=1}^{T} \delta_i t_i s_i + \sum_{i=T+1}^{p} \delta_i t_i s_i \\ \text{s.t.}\quad \varphi_i(x) - t_i \ge 0,\ i = 1, 2, \ldots, T, \\ \qquad \varphi_i(x) - t_i \le 0,\ i = T+1, T+2, \ldots, p, \\ \qquad s_i \psi_i(x) \le 1,\ i = 1, 2, \ldots, T, \\ \qquad s_i \psi_i(x) \ge 1,\ i = T+1, T+2, \ldots, p, \\ \qquad x \in D,\ (t, s) \in \Omega^0. \end{cases}$$

The key equivalent theorem for the problems (SLR) and (EQ) is given as follows.

Theorem 1. *If (x^*, t^*, s^*) is a global optimum solution of the problem* (EQ)*, then we can get that $t_i^* = \varphi_i(x^*)$, $s_i^* = 1/\psi_i(x^*)$, $i = 1, 2, \ldots, p$, and x^* is a global optimum solution of the problem* (SLR)*. On the contrary, if x^* is a global optimum solution of the problem* (SLR)*, define $t_i^* = \varphi_i(x^*)$ and $s_i^* = 1/\psi_i(x^*)$ $(i = 1, 2, \ldots, p)$, then (x^*, t^*, s^*) is a global optimum solution of the problem* (EQ)*.*

Proof. The proof can be easily followed by the monotonicity of the function, therefore it is omitted here. □

By Theorem 1, we can follow that, for solving the problem (SLR), we may globally solve its equivalent problem (EQ) instead. Besides, it is easy to understand that the problems (SLR) and (EQ) have the same global optimal value.

2.2. Linear relaxation programming

In this subsection, we will establish the linear relaxation programming of the problem (EQ), which can offer the upper bound of the global optimal value for the (EQ) in the proposed branch and bound algorithm. By utilizing special structure of objective function and constraint functions for the (EQ), the proposed method for yielding this linear relaxation programming problem depends on overestimating or underestimating the objective function and constraint functions of the (EQ). By Horst and Tuy (1993), McCormick (1976), Tuy (1998) and Benson (2004b), we can propose an approach for generating this linear overestimating function of the objective function for the problem (EQ), which is given by the following Theorem 2.

Theorem 2. *Consider the rectangle $LR = \{(t, s) \in R^2 \mid l \le t \le u,\ L \le s \le U\}$, where $l,\ u,\ L$ and U are all non-negative constants satisfying $0 \le l \le u,\ 0 < L \le U$. For any $(t, s) \in LR$, define the functions $g(t, s),\ g^{LR}(t, s)$ and $g_{LR}(t, s)$ as follows:*

$$g(t, s) = ts,$$
$$g^{LR}(t, s) = \min\{Ut + ls - lU, Lt + us - uL\},$$
$$g_{LR}(t, s) = \max\{Lt + ls - lL, Ut + us - uU\}.$$

Then, the following conclusions hold:

(i) For any $(t, s) \in LR$, $g(t, s), g^{LR}(t, s)$ and $g_{LR}(t, s)$ satisfy

$$g_{LR}(t, s) \le g(t, s) \le g^{LR}(t, s).$$

(ii) Let $\Delta t = u - l$, $\Delta s = U - L$, then

$$\lim_{\Delta s \to 0} g_{LR}(t, s) = \lim_{\Delta s \to 0} g(t, s) = \lim_{\Delta s \to 0} g^{LR}(t, s).$$

(iii) Let $LR = \{(t, s) \in R^2 \mid l \le t \le u, s = b\}$, then we have

$$g^{LR}(t, s) = g(t, s) = g_{LR}(t, s).$$