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Discrete Optimization

Characterization of the split closure via geometric lifting

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ABSTRACT

We analyze *split cuts* from the perspective of *cut generating functions* via *geometric lifting*. We show that α-cuts, a natural higher-dimensional generalization of the k-cuts of Cornuéjols et al., give all the split cuts for the mixed-integer corner relaxation. As an immediate consequence we obtain that the k-cuts are equivalent to split cuts for the 1-row mixed-integer relaxation. Further, we show that split cuts for finite-dimensional corner relaxations are restrictions of split cuts for the infinite-dimensional relaxation. In a final application of this equivalence, we exhibit a family of pure-integer programs whose split closure has arbitrarily bad integrality gap. This complements the mixed-integer example provided by Basu et al. (2011).

1. Introduction

The existing literature on cutting planes for general integer programs can be roughly partitioned into two classes. One class of cutting planes relies on the paradigm of *cut-generating functions* (CGFs) (Conforti, Cornuéjols, Daniilidis, Lemaréchal, & Jérôme, 2013), whose roots can be traced back to Gomory and Johnson's work in the 1970s. Prominent examples in this class are *Gomory mixed-integer cuts (GMI), mixed-integer rounding cuts (MIR), reduce-and-split cuts* and *k-cuts* (Andersen, Cornuéjols, & Li, 2005a; Cornuéjols, Li, & Vandenbussche, 2003; Gomory, 1963; Nemhauser & Wolsey, 1990). The other class of cutting planes is based on the idea taking convex hulls of disjunctions; this perspective goes back to Balas' work on *disjunctive programming* (Balas, 1979). Examples in this class are *split cuts, lift-and-project cuts* and, more recently, *t-branch disjunctions* (Balas, Ceria, & Cornuéjols, 1993; Dash & Günlük, 2013; Sherali & Adams, 1990).

One important advantage of the cut-generating function paradigm is that often they give closed form formulas to generate cuts that can be computed very efficiently. In contrast, the disjunctive programming paradigm usually involves solving a *cut generating linear program* (Balas et al., 1993), which can be computationally expensive. This leads to a significant blowup in time required to generate cuts using the disjunctive paradigm, as opposed to the cut generating function framework. Hence, there is great value in results which show that a certain family of disjunctive cuts can also be obtained by cut generating functions that are computable efficiently. Moreover, establishing connections between the cut-generating and the disjunctive paradigms contributes to the overarching goal of obtaining a more complete understanding of cutting planes.

In this paper, we analyze the family of *split cuts* (Cook, Kannan, & Schrijver, 1990) from this viewpoint. Split cuts are one of the most important cuts in practice and their effective use was responsible for a major improvement in MIP solvers since the 1990s (Achterberg & Wunderling, 2013; Bixby, Fenelon, Gu, Rothberg, & Wunderling, 2004). Due to their importance, these cuts have been extensively studied, both theoretically (Andersen, Cornuéjols, & Li, 2005b; Basu, Cornuéjols, & Margot, 2012; Nemhauser & Wolsey, 1990) and computationally (Balas, Ceria, Cornuéjols, & Natraj, 1996; Balas & Saxena, 2008; Bonami, 2012; Dash & Goycoolea, 2010; Fischetti & Salvagnin, 2011, 2013).

An important tool for the cut generating function approach has been the study of finite and infinite dimensional *corner relaxations*. Corner relaxations retain enough of the complexity of mixed-integer programs to be extremely useful models for obtaining general cutting planes, and yet have a structure that yields to mathematical analysis and a beautiful theory has been built about them (Conforti, Cornuéjols, & Zambelli, 2011a).

In Section 1.1, we formally define the main objects of study of this paper, and then follow up with a statement of our results in Section 1.2.

1.1. Preliminaries

1.1.1. Corner relaxations

Given a point $f \in [0, 1]^n \setminus \mathbb{Z}^n$ and sets $R, Q \subseteq \mathbb{R}^n$ (not necessarily finite), define the *(n-row) generalized corner relaxation* $\mathcal{C} = \mathcal{C}(f, R, Q)$

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as the set solutions (x, s, y) to the system

$$\begin{aligned} x &= f + \sum_{r \in R} r \cdot s(r) + \sum_{q \in Q} q \cdot y(q) \\ x &\in \mathbb{Z}^n \qquad\qquad\qquad\qquad\qquad\qquad (\mathcal{C}(f,R,Q)) \\ y(q) &\in \mathbb{Z} \quad \forall q \in Q \\ s &\in \mathbb{R}_+^{(R)}, y \in \mathbb{R}_+^{(Q)}, \end{aligned}$$

where $\mathbb{R}_+^{(S)}$ denotes the set of non-negative functions $f : S \to \mathbb{R}_+$ with finite support. This model was first introduced by Johnson (1974). When R and Q are finite subsets of $\mathbb{R}^n$, the above problem is referred to as the *mixed-integer corner polyhedron*.

We define the *continuous corner relaxation* $\mathcal{CC}(f, R, Q)$ obtained from ($\mathcal{C}(f, R, Q)$) by dropping the integrality constraint on the y variables, i.e., all points (x, s, y) that satisfy the first, second and fourth constraints. We use $\mathcal{C}_{LP}(f, R, Q)$ to denote the linear relaxation of $\mathcal{C}(f, R, Q)$, namely the set of points (x, s, y) that satisfy the first and fourth set of constraints in ($\mathcal{C}(f, R, Q)$). Thus we have that $\mathcal{C}(f, R, Q) \subseteq \mathcal{CC}(f, R, Q) \subseteq \mathcal{C}_{LP}(f, R, Q)$.

1.1.2. Valid cuts

A *valid cut* or *valid function* for $\mathcal{C}(f, R, Q)$ (or $\mathcal{CC}(f, R, Q)$) is a pair of non-negative functions $(\psi, \pi) \in \mathbb{R}_+^R \times \mathbb{R}_+^Q$ (where $\mathbb{R}_+^S$ denotes the set of functions from S to $\mathbb{R}_+$) such that for every $(x, s, y) \in \mathcal{C}(f, R, Q)$ (or $(x, s, y) \in \mathcal{CC}(f, R, Q)$ respectively) we have

$$\sum_{r \in R} \psi(r)s(r) + \sum_{q \in Q} \pi(q)y(q) \geq 1. \qquad (1)$$

We will associate the function pair (ψ, π) with the cut defined by the above inequality. With slight abuse of notation, we may use (ψ, π) to denote the set of points satisfying (1) (e.g., $\mathcal{C}_{LP}(f, R, Q) \cap (\psi, \pi)$).

Also with slight overload in notation, given sets $R' \supseteq R$ and $Q' \supseteq Q$ and functions $\psi \in \mathbb{R}^{R'}$ and $\pi \in \mathbb{R}^{Q'}$, we say that (ψ, π) is a valid cut/function for $\mathcal{C}(f, R, Q)$ if the restriction $(\psi|_R, \pi|_Q)$ is valid for it. Notice that if (ψ, π) is valid for $\mathcal{C}(f, R', Q')$, then it is valid for the restriction $\mathcal{C}(f, R, Q)$.

1.1.3. GMI and α-cuts

We define a family of cut-generating functions for the n-row corner relaxation, that we call *α-cuts*, which is a natural higher-dimensional generalization of k-cuts (Cornuéjols et al., 2003). Informally, an α-cut for an n-row corner relaxation is obtained by taking an integer vector $\alpha \in \mathbb{Z}^n$, aggregating the n rows of the problem using the α's as multipliers and then employing the GMI function to the resulting equality.

Given a real number $a \in \mathbb{R}$, let $[a]$ denote its fractional part $a - \lfloor a \rfloor$. Then given $f \in \mathbb{R}$, the *GMI function* $(\psi^f_{\text{GMI}}, \pi^f_{\text{GMI}})$ is defined as

$$\begin{aligned} \psi^f_{\text{GMI}}(r) &= \max\left\{\frac{r}{1-[f]}, -\frac{r}{[f]}\right\}, \\ \pi^f_{\text{GMI}}(q) &= \max\left\{\frac{[q]}{1-[f]}, \frac{1-[q]}{[f]}\right\}. \end{aligned} \qquad (2)$$

The GMI function is valid for the infinite corner relaxation $\mathcal{C}(f, \mathbb{R}, \mathbb{R})$ $(n = 1)$.

For $f \in [0, 1]^n \setminus \mathbb{Z}^n$, define the sets $\mathbb{Z}_f = \{w \in \mathbb{Z}^n : fw \notin \mathbb{Z}\}$. Given $\alpha \in \mathbb{Z}_f$, we define the *α-cut function* $(\psi^f_\alpha, \pi^f_\alpha)$ is defined as

$$\psi^f_\alpha(r) = \psi^{\alpha f}_{\text{GMI}}(\alpha r), \qquad \pi^f_\alpha(q) = \pi^{\alpha f}_{\text{GMI}}(\alpha q). \qquad (3)$$

We make the observation here that $\psi^{-f}_{\text{GMI}}(-r) = \psi^f_{\text{GMI}}(r)$ and similarly, $\pi^{-f}_{\text{GMI}}(-q) = \pi^f_{\text{GMI}}(q)$ (since $[-a] = 1 - [a]$). Therefore, $\psi^f_{-\alpha} = \psi^f_\alpha$ and $\pi^f_{-\alpha}(q) = \pi^f_\alpha(q)$, and one needs only consider $\alpha \in \mathbb{Z}_f$ lying on one side of any halfspace in $\mathbb{R}^n$. In the case $n = 1$, the α-cut functions correspond to the *k-cuts* introduced in Cornuéjols et al. (2003).

1.1.4. Split cuts

Consider an n-dimensional corner relaxation $\mathcal{C}(f, R, Q)$. Given $\alpha \in \mathbb{Z}_f$ and $\beta : Q \to \mathbb{Z}$, we define the split disjunction

$$D(\alpha, \beta, f) \triangleq \left\{(x, s, y) : \alpha x + \sum_{q \in Q} \beta(q)y(q) \leq \lfloor \alpha f \rfloor\right\} \cup \left\{(x, s, y) : \alpha x + \sum_{q \in Q} \beta(q)y(q) \geq \lceil \alpha f \rceil\right\}.$$

A cut (ψ, π) is a *split cut* for $\mathcal{C}(f, R, Q)$ with respect to the disjunction $D(\alpha, \beta, f)$ if it is satisfied by all points in the set $\mathcal{C}_{LP}(f, R, Q) \cap D(\alpha, \beta, f)$.[1]

We analogously define split cuts for $\mathcal{CC}(f, R, Q)$. For that, given $\alpha \in \mathbb{Z}_f$, define the split disjunction

$$\bar{D}(\alpha, f) = \{(x, s, y) : \alpha x \leq \lfloor \alpha f \rfloor\} \cup \{(x, s, y) : \alpha x \geq \lceil \alpha f \rceil\}.$$

The cut (ψ, π) is a *split cut* for $\mathcal{CC}(f, R, Q)$ with respect to the disjunction $\bar{D}(\alpha, f)$ if it is satisfied by all points in $\mathcal{C}_{LP}(f, R, Q) \cap \bar{D}(\alpha, f)$.

Notice that these definitions hold for the case where R and/or Q are infinite, thus generalizing the standard definition of split cuts to this setting.

1.2. Statement of results

We show that the α-cuts give all the split cuts for a generalized mixed-integer corner relaxation, which includes both the finite and infinite dimensional relaxations.

Theorem 1. *Consider an n-row generalized corner relaxation $\mathcal{C}$. Then the α-cuts are exactly the split cuts for $\mathcal{C}$. More precisely, every α-cut is a split cut for $\mathcal{C}$ and every split cut for $\mathcal{C}$ is dominated by an α-cut.*

In particular, this result directly gives the equivalence between k-cuts and split cuts for the 1-row (finite or infinite dimensional) corner relaxations.

The proof of Theorem 1 is based on *geometric lifting*, a technique introduced in Conforti, Cornuéjols, and Zambelli (2011b). The cut-generating functions that we obtain can also be derived using the formulas from Section 2.4 of Andersen et al. (2005a). However, in our opinion, our geometric viewpoint illuminates aspects of these cut generating functions that are not immediately apparent if one uses the algebraic approach of Andersen et al. (2005a). We illustrate this advantage using two applications: Theorems 2 and 3 below.

It is known that every cut for the finite-dimensional corner relaxation is a restriction of a cut for the infinite relaxation. We show that, not only does the infinite relaxation encode all finite-dimensional corner relaxations and valid cuts for them, but it also preserves the structure of split cuts.

Theorem 2. *Consider a non-empty n-row generalized corner relaxation $\mathcal{C}(f, R, Q)$. Then a valid cut for $\mathcal{C}(f, R, Q)$ is a split cut if and only if it is the restriction of a split cut for the infinite relaxation $\mathcal{C}(f, \mathbb{R}^n, \mathbb{R}^n)$.*

This result relies on the fact that the characterization of Theorem 1 holds for the infinite relaxation.

In addition, we use insights obtained from this characterization to better understand the strength of split cuts. Cook et al. (1990) constructed an example of a *mixed-integer* program with infinite split rank. Reinforcing the potential weakness of the split closure, Basu, Bonami, Cornuéjols, and Margot (2011) constructed *mixed-integer* programs whose split closures provide an arbitrarily weak approximation to their integer hull. On the other hand, it is known that every *pure-integer* program (where all variables are integer valued)

[1] Notice that we only consider α in $\mathbb{Z}_f$, and not in $\mathbb{Z}^n$, because in the latter case $\mathcal{C}_{LP}(f, R, Q) \cap D(\alpha, \beta, f) = \mathcal{C}_{LP}(f, R, Q)$.