European Journal of Operational Research 243 (2015) 912–920

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

A solution concept for network games: The role of multilateral interactions

Surajit Borkotokey[a,*], Rajnish Kumar[b], Sudipta Sarangi[c]

[a] Department of Mathematics, Dibrugarh University,Dibrugarh, India
[b] Department of Economics, Queen's University Management School, Queen's University Belfast, Belfast, UK
[c] DIW Berlin and Department of Economics, Louisiana State University, LA, USA

ARTICLE INFO

Article history:
Received 8 November 2013
Accepted 15 December 2014
Available online 26 December 2014

Keywords:
Network games
Allocation rules
Cooperative games

ABSTRACT

We propose an allocation rule that takes into account the importance of both players and their links and characterize it for a fixed network. Our characterization is along the lines of the characterization of the Position value for Network games by van den Nouweland and Slikker (2012). The allocation rule so defined admits multilateral interactions among the players through their links which distinguishes it from the other existing rules. Next, we extend our allocation rule to flexible networks à la Jackson (2005).

1. Introduction

The study of networks under the game theoretic framework revolves around two basic problems: how a network is formed and how to allocate the value it generates among its members. In this paper we propose a new allocation rule that takes into account the importance of players as well as their links. Since a network describes the interaction structure between agents, a key feature of our allocation rule is that it covers both bilateral and multilateral interactions. We provide an axiomatic characterization of this rule and compare it to other allocation rules in the literature.

The notion of a "Network game" under the cooperative framework was introduced by Jackson and Wolinsky (1996). This itself is an extension of the "Communication situation" concept introduced by Myerson (1977). In a subsequent paper Jackson (2005) extended a solution concept for Communication situations to Network games: the Myerson value and called it "the equal bargaining rule". Borm, Owen, and Tijs (1992), Slikker and van den Nouweland (2001), Slikker (2005, 2007), Haeringer (2006), Kamijo (2009), Kamijo and Kongo (2009), Ghintran (2013), van den Nouweland and Slikker (2012) etc., extended and characterized another solution concept for Communication situations and Network games: the Position value. Recently the kappa value is defined by Belau (2013) as a more generalized version of the position value for Communication situations that accounts for various potential alternatives to the actual network in place. Related works by Bergantiños, Gómez-Rúa, Llorca, Pulido, and Sánchez-Soriano (2014) and Rosenthal (2013) study the division of cost of a network among the agents.

Jackson (2005) introduced the notion of flexible networks. The underlying idea is that the network observed at a point in time need not be fixed, and could ultimately evolve into a better subnetwork if such a network exists. He showed that the equal bargaining rule for Network games possesses certain limitations. In fact similar arguments can also be constructed for some of the characterizations of the Position value for Network games. Jackson proposed another set of allocation rules for flexible networks among which, the Player Based Flexible Network allocation rule (PBFN) and the Link Based Flexible Network allocation rules (LBFN) are quite appealing. While flexible network allocation rules have many advantages, they may not always be feasible.[1]

The Position value is a standard link based rule where players get less importance than the links they make. The equal bargaining rule on the other hand emphasizes more on players. Both these values are Shapley type in the sense that they aggregate the marginal contributions of the players over links and coalitions stemming from the given network respectively. The Position value aggregates the marginal contributions of the links and then divide this value into equal halves between the two players forming the links. The cumulative value arising from the links of a player is her final payoff. This shows the importance of the roles links play here. On the other hand the equal bargaining rule aggregates the marginal contributions of the players involved in

* Corresponding author. Tel.: +91 9954430677; fax: +91 373 2370323.
E-mail address: sborkotokey@dibru.ac.in (S. Borkotokey).

[1] For instance there might be situations where a priori assessment of the values of alternative networks is not possible due to the need for costly hardware installations.

http://dx.doi.org/10.1016/j.ejor.2014.12.027
0377-2217/

each coalition restricted to the network. Thus the contribution of a player in this setup is measured by her joining or leaving the network with all her links. Therefore these two values are two extreme cases of interactions of a player with her peers. However, it is useful to have an allocation rule that accounts for both players and their links in a manner that incorporates the simultaneous multilateral interactions in a network. By multilateral interactions, we mean the interactions that take place among individuals through some (possibly more than one) of their links simultaneously in the network. For example, consider players 1, 2, and 3 of a particular football team who complement each other on the field. Suppose these players are linked to each other (and the rest of the team) by means of their passing game which ultimately leads to goals being scored. Since player 1 is involved in two links with 2 and 3 respectively and complementarities exist across players, this player's contribution to the team will change depending on whether we consider the removal of those links separately or the removal of both the links together. Thus, measuring the marginal contributions of one of those players cannot be done independently from the others and the effect of the multilateral interaction needs to be considered.

In this paper, we develop such an allocation rule for fixed network games and extend it to flexible networks. First we provide an axiomatic characterizations of our rule using standard Shapley like axioms viz., *linearity*, *null player* axiom, *anonymity*, *monotonicity* and *efficiency* suitable for the standard revenue or cost allocation problems. We then show that this is satisfied only for a small set of networks.

On the other hand, cooperative game theory also had a long tradition of offering inefficient solution concepts. There are many situations like creating a public good or winning a voting situation where an allocation is not just to divide the winnings but to describe how the players can affect the outcome of the game through their interactions.[2] In Owen (1975), a modification of the Banzhaf value is defined for communication situations. This was characterized in Alonso-Meijide and Fiestras-Janeiro (2006) using the properties of component total power and fairness. Inefficient allocation rules are likely to describe the prospects of playing different roles in a game rather than describing fair division (see Dubey, Neyman, & Weber, 1981). In networks, when players have multilateral interactions with their peers, they have different roles to play in each such interaction. Thus for an allocation rule that measures the prospects of playing those roles (interacting with players at different levels) axioms like *efficiency* can be replaced by weaker axioms. Keeping this in mind, next we replace the axiom of *efficiency* by a new axiom, called the axiom of *multilateral interaction* and obtain an alternative characterization.

We show that our allocation rule can provide a better measure of a player's prospects in the network and determines the total power (viz., power to initiate an action or power to prevent an action) of a player in the network due to her multilateral interactions. Here power of a player is measured by her prospects in playing different roles. These roles are interactions among players at different levels. In either case (with or without *efficiency*), we obtain a Shapley like allocation rule which finds the weighted average of the marginal contributions of players taking into account every possible combination of their links. Hence while the player based approaches account for contributions of all links of a player and the link based approaches for those of one link of each player, our rule accounts for everything in between.

[2] This idea is similar to most of the semi-values for cooperative games (e.g., the minimum norm solutions, Kultti & Salonen, 2007; least square values, Ruiz, Valenciano, & Zarzuelo, 1996, 1998; Banzhaf value, Banzhaf, 1965 for simple games and their extensions to general cooperative games i.e., TU games by Owen, 1975; Dragan, 1996; Marichal & Mathonet, 2011 etc.) and may be thought of as describing the power of players.

In the literature, excepting van den Nouweland and Slikker (2012), both the equal bargaining rule and Position value for Network games are characterized by the rather strong axiom of component balance. It requires that when the Network game does not allow spill-over across components (alternatively: the game is component additive), the value generated by any component should be allocated to its constituent players. We do not require component balancedness for our characterizations.

The paper proceeds as follows. Section 2 provides the necessary mathematical preliminaries, includes a brief description of the existing allocation rules and their characterization results. Section 3 develops our notion of an interactive allocation rule and its characterization. In Section 4 we provide numerical solutions to the motivating examples. Section 5 concludes. Major proofs and the additional details about Example 1 can be found in Appendix A.

2. Preliminaries

In this section, we present the definitions and results required for development of our model.

Let $N = \{1, \ldots, n\}$ be a fixed set of players who are connected in some network relationship. A network consists of a finite set of elements called nodes corresponding to players and a finite set of pairs of nodes called links which correspond to bilateral relationships between players. The network g is thus a list of unordered pairs of players $\{i, j\}$, where $\{i, j\} \in g$ indicates that i and j are linked in the network g. For simplicity, we write ij to represent the link $\{i, j\}$. The degree of a player in a network is the number of direct links she has in the network. Let g^N be the set of all subsets of N of size 2. We call g^N the complete network with n nodes. Then $G = \{g | g \subseteq g^N\}$ denotes the set of all possible networks or graphs on N. Special types of networks that are important for us are k-regular networks (network where every node has the same degree k) and the star networks (networks in which $n-1$ peripheral players are connected through one central player). The network obtained by adding another network g' to an existing network g is denoted by $g + g'$ and the network obtained by deleting subnetwork g' from an existing network g is denoted by $g - g'$. For $g \in G$, $l(g)$ denotes the total number of links. By $m \in g$ we mean that m is a subnetwork of g ($m \subseteq g$) with length $l(m) = 1$. A path in a network is a sequence of nodes such that from each of its nodes there is a link to the next node in the sequence. Let $N(g)$ be the set of players who have at least one link in g. That is,

$$N(g) = \{i \mid \exists j \text{ such that } ij \in g\}$$

Let $n(g) = \#N(g)$ be the number of players involved in g. Let $L_i(g)$ be the set of links that player i is involved in, so that

$$L_i(g) = \{ij \mid \exists\, j \ :\ ij \in g\},$$

We denote by $l_i(g)$ the number of links in player i's link set. It follows that $l(g) = \frac{1}{2}\sum_i l_i(g)$. Given any $S \subseteq N$, let g^S be the set of all subsets of S of size 2, i.e., the complete network formed by the players in S. Let $g|_S$ denote the subnetwork of g formed by the players in S. Formally we have,

$$g|_S = \{ij \mid ij \in g \text{ and } i \in S, j \in S\}.$$

Definition 1. A component of a network g, is a non-empty subnetwork $g' \subset g$, such that

(a) if $i \in N(g')$ and $j \in N(g')$ where $j \neq i$, then there is a path in g' between i and j, and
(b) if $i \in N(g')$ and $ij \in g$, then $ij \in g'$.

Essentially, components of a network are the distinct connected subgraphs in it. The set of components of g is denoted by $C(g)$.