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Decision Support

The analytic hierarchy process with stochastic judgements

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ABSTRACT

The analytic hierarchy process (AHP) is a widely-used method for multicriteria decision support based on the hierarchical decomposition of objectives, evaluation of preferences through pairwise comparisons, and a subsequent aggregation into global evaluations. The current paper integrates the AHP with stochastic multicriteria acceptability analysis (SMAA), an inverse-preference method, to allow the pairwise comparisons to be uncertain. A simulation experiment is used to assess how the consistency of judgements and the ability of the SMAA-AHP model to discern the best alternative deteriorates as uncertainty increases. Across a range of simulated problems results indicate that, according to conventional benchmarks, judgements are likely to remain consistent unless uncertainty is severe, but that the presence of uncertainty in almost any degree is sufficient to make the choice of best alternative unclear.

1. Introduction

The analytic hierarchy process (Saaty, 1990) is a widely-used method for multicriteria decision support based on a hierarchical decomposition of a decision problem into multiple criteria, the assessment of preferences using pairwise comparisons, and an aggregation of these pairwise preferences into an overall evaluation of the alternatives. While a number of practical and theoretical aspects of the AHP have proved controversial (see for example the discussion in Belton & Stewart (2002)), it has found widespread application and acceptance in practice (e.g. Vaidya & Kumar, 2006), to the extent that it may well be among the most-frequently applied of currently available methods for decision support.

At the heart of the method is a nine-point semantic scale used by decision makers to express their preferences for one alternative over another on a particular criterion, and for how much one criterion is valued over another. It is clear that sometimes these assessments will be subject to uncertainty – meaning that the decision maker (DM) does not possess the necessary information to describe or deterministically predict the inputs required by the AHP (see Durbach & Stewart (2012) for a review of uncertainty in multicriteria decision support). Although the standard AHP method does not directly treat uncertainty or imprecision in its inputs, a number of extensions have been proposed to address this issue, using for example fuzzy set theory (Boender, de Graan, & Lootsma, 1989; Buckley, 1985; Laarhoven & Pedrycz, 1983), interval arithmetics (Salo & Hämäläinen, 1995), and various stochastic techniques (Hauser & Tadikamalla, 1996; Saaty & Vargas, 1987).

This paper adds to that body of work by introducing a simulation-based method for representing imprecise or uncertain pairwise comparison information from one or more DMs through stochastic distributions, and a computational method to treat this information in the analysis. The method is a variant of stochastic multicriteria acceptability analysis (SMAA; see Lahdelma, Hokkanen, & Salminen (1998), Lahdelma & Salminen (2001), Tervonen, Hakonen, & Lahdelma (2008)), an inverse-preference methodology applied here to the case of the AHP. The resulting SMAA-AHP can be used with arbitrary independent or dependent distributions for the comparisons, and is based on Monte Carlo simulation from probability distributions appropriately defined over any uncertain pairwise comparisons and a subsequent collection of statistics summarizing the performance of each alternative. SMAA-AHP is related to other simulation-based methods, most notably Hauser and Tadikamalla (1996), but presents additional information to the DM, defines uncertainty regions differently, and uses a different distribution for the uncertain judgements. SMAA-AHP also allows more flexible representation of weight constraints and can also be used with missing preference information.

The remainder of the paper is organized as follows. Section 2 reviews uncertainty modelling in the AHP. Section 3 describes

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the SMAA-AHP method. Section 4 demonstrates the method using a small example. Section 5 discusses the advantages and potential problems with the method, guided by the results of a simulation study. A final section concludes the paper.

2. Uncertainty modelling in the AHP

In the following, we consider a decision problem consisting of I alternatives, each evaluated on K criteria. Let z_{ik} be the evaluation of alternative i in terms of criterion k, according to some suitable performance measure. In the standard AHP the DM performs pairwise comparisons at each node of the objectives hierarchy, expressing their preferences for one alternative over another on a particular criterion, or for how much one criterion is valued over another. The pairwise preference a_{ijk} for alternative i over alternative j on criterion k represents the ratio between evaluations z_{ik}/z_{jk}, expressed on a discrete scale from 1 to 9 (where 1 means equal preference and 9 denotes absolute preference). Where convenient, we drop the criterion subscript and refer simply to the pairwise evaluation a_{ij}. The same approach is used to compare the importance of criteria, in which case we refer to a pairwise preference a_{ij} for criterion i over criterion j representing the ratio between trade-off weights w_i/w_j. In cases where pairwise comparisons can be assessed precisely, a number of ways have been proposed to aggregate these into global measures of performance (Belton & Stewart, 2002). Most commonly, the eigenvector corresponding to the largest eigenvalue of the ($I \times I$ or $K \times K$) pairwise comparison matrix $\mathbf{A} = [a_{ij}]$ is extracted (the so-called priority vector), and a global evaluation formed by a simple weighted sum.

Our concern is with decision making situations in which the pairwise evaluations a_{ijk} (and consequently computed values for z_{ik} and w_j) are uncertain. Early research into the modelling of probabilities in the AHP was largely concerned with deriving relationships between the distributional form of the uncertain pairwise judgements and the distributions of the marginal evaluations contained in the priority vector (Basak, 1989, 1991; Saaty & Vargas, 1987; Vargas, 1982). Subsequent probabilistic AHP models (Banuelas & Antony, 2007; Basak, 1998; Hauser & Tadikamalla, 1996; Levary & Wan, 1998; Levary & Wan, 1999) have focused on using Monte Carlo simulation to randomly generate pairwise evaluations from the distributions specified by decision makers. These approaches all follow the same basic approach, first expressed by Hauser and Tadikamalla (1996). The decision maker expresses pairwise comparisons in the usual way i.e. using the same 1–9 scale as for deterministic AHP, except that these comparisons are allowed to be random variables with associated probability distributions. Hauser and Tadikamalla generated random judgements a^*_{ij} uniformly on the interval $[a_{ij} - da_{ij}, a_{ij} - da_{ij}]$, with d an uncertainty factor, before transforming any values less than one using $f(a^*_{ij}) = 1/(2 - a^*_{ij})$. Further restrictions may be placed on the types of distributions if necessary. Next sets of random pairwise judgements are generated using Monte Carlo simulation. For each set of randomly generated evaluation matrices the priority vector is computed. Repeating this process many times gives a distribution of priorities for each alternative, which can be used to rank the alternatives, in most cases using the mean of the distribution.

Most authors make small embellishments around this general process. Levary and Wan (1998) incorporate scenarios into their model (see also Levary & Wan (1999)). Decision makers thus assess different (possibly stochastic) judgemental matrices for each scenario. Their simulation approach first generates a random number to specify which scenario is being used, and then generates further random numbers specifying the pairwise judgements within each scenario. Basak (1998) uses a Bayesian approach to integrate expert judgements with the decision maker's prior probabilistic assessments. Pairwise judgements are simulated by drawing from the posterior distributions. Banuelas and Antony (2007) add a sensitivity analysis phase to investigate the influence of the probabilistic judgements on the consistency index. As mentioned above the primary distinction between existing simulation-based AHP methods and SMAA-AHP is the additional information that is presented to DMs, which can be useful in facilitating a greater understanding of the decision problem and progressing towards a final decision. We discuss this information in the presentation of the SMAA-AHP given in the following section.

3. The SMAA-AHP method

In SMAA-AHP, the DMs may express their comparisons on a discrete scale from 1 to 9 or use arbitrary positive values. The DMs can give their pairwise comparisons either as precise values, as in AHP, or as intervals to express imprecise or uncertain preferences. The DMs can give the lower and upper bounds of the intervals explicitly, or express them as $[a_{ij}/d_{ij}, a_{ij}d_{ij}]$ where a_{ij} is the geometric mean of the interval and $d_{ij} \geqslant 1$ is the so-called imprecision factor of their pairwise comparison. For example, the interval $[0.5, 8]$ corresponds to the pairwise comparison 2 with imprecision factor 4. The imprecision factor is a meaningful way to express uncertainty on a ratio scale, where all values should be positive.

When the DMs express their pairwise comparisons, it should be checked that these are sufficiently consistent. In the original AHP, where pairwise comparisons are expressed deterministically, a popular approach for evaluating consistency is to compare λ_1, the leading eigenvalue of an assessed pairwise comparison matrix, with I, the leading eigenvalue obtained from an $I \times I$ matrix of perfectly consistent judgements (in the sense that $a_{ik} = a_{ij}a_{jk}, \forall i, j, k$). To provide a measure of the severity of this deviation, $(\lambda_1 - I)/(I - 1)$ is compared with the mean inconsistency value derived from many randomly generated reciprocal matrices of the same size. An inconsistency ratio of 0.1 or less is generally stated to be acceptable (Saaty, 1990), meaning that the inconsistency of the observed pairwise comparisons should be no more than 10% of what would be observed, on average, from completely random judgements. For an interval-based analysis, a natural analogue would be to suggest that the geometric mean of the comparisons should have a inconsistency ratio below 10%. Note, however, that this benchmark, as well as the general use of the inconsistency ratio, has been strongly criticized (see in particular Bana e Costa & Vansnick (2008)).

After each DM has given his/her pairwise comparisons, we combine them into intervals $[a^{\min}_{ij}, a^{\max}_{ij}]$ where $a^{\min}_{ij}$ is the minimal value that any DM has expressed and $a^{\max}_{ij}$ is the maximal value. We represent then the aggregated comparison values by stochastic variables with suitable probability distributions. Technically, it is possible to use arbitrary distributions. However, in the absence of information about the distribution, we apply the truncated and scaled $1/x$ distribution. The PDF (probability density function) of the scaled $1/x$ distribution is given by $f(x) = \alpha/x$ when $x \in [x^{\min}, x^{\max}]$, and zero elsewhere. The scaling coefficient $\alpha = 1/\ln(x^{\max} - x^{\min})$ is determined so that the integral over the PDF equals one.

The motivation for using the scaled $1/x$ distribution to represent pairwise comparisons in an interval is that this distribution allocates equal probability mass for all sub-intervals $[x/d, xd]$ corresponding to the same imprecision factor d. For example, given a pairwise comparison interval $[0.5, 8]$, the scaled $1/x$ distribution allocates equal probability mass of 1/4 for each of the subintervals $[0.5, 1], [1, 2], [2, 4]$, and $[4, 8]$. If the interval is degenerate, i.e. $a^{\min}_{ij} = a^{\max}_{ij}$, we use Dirac's delta function (the unit impulse function) as the distribution.