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Decision Support

Fast and fine quickest path algorithm

Antonio Sedeño-Noda *, Jonathan D. González-Barrera

Departamento de Estadística, Investigación Operativa y Computación (DEIOC), Universidad de La Laguna, San Cristóbal de La Laguna, CP 38271 Santa cruz de Tenerife, Spain

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ABSTRACT

We address the quickest path problem proposing a new algorithm based on the fact that its optimal solution corresponds to a supported non-dominated point in the objective space of the minsum–maxmin bicriteria path problem. This result allows us to design a label setting algorithm which improves all existing algorithms in the state-of-the-art, as it is shown in the extensive experiments carried out considering synthetic and real networks.

1. Introduction

The quickest path problem (QPP) consists of finding a path in a directed network to transmit a given amount of data σ between a source node s and a destination node t with minimum transmission time. The transmission time depends on two parameters: an additive function that the represents the path delay, and a bottleneck function that represents the path capacity or bandwidth. The QPP arises in several applications in road transportations, telecommunication networks, convoy movement problems, etc.

The QPP was first proposed by Moore (1976), who gave an algorithm running in $O(hm + hn\log n)$ time, where n and m are the number of nodes and arcs, respectively, in the network, and h is a parameter smaller than the number of different capacities greater than the capacity of the shortest path with respect to lead time. Since then, several algorithms solving QPP appeared in the literature. The algorithm of Chen and Chin (1990) transforms the network (r copies of the original network) and then compute shortest paths according to lead time, selecting finally one with the lowest transmission time. The algorithm runs in $O(rm + rn\log rn)$ time and uses $O(r(n + m))$ space for a given value of σ, where $r \leqslant m$ is the number of distinct capacities in the original network. Rosen et al. (1991) developed an alternative algorithm solving the QPP by a sequence of shortest path computations on networks where the minimum capacity increases. They obtain an algorithm running in $O(rm + rn\log n)$ time, but using $O(n + m)$ space only.

Martins and Santos (1997) interpreted the QPP as a bicriteria path problem computing all non-dominated points in the objective space, and finally selecting the non-dominated path with minimum transmission time. The complexity of this algorithm matches the complexity of the algorithm by Rosen et al. (1991).

However, Park et al. (2004) have pointed out two drawbacks in the previous algorithms. The first concern is that the existing algorithms must enumerate all non-dominated paths with different capacities. Besides, the σ value is only used to finally solve the QPP. On the other hand, a Dijkstra (1959) labeling algorithm might fail to solve QPP because this problem does not satisfy the optimality principle of dynamic programming. Thus, they design a label-setting algorithm transforming in an implicit way the original network such that any sub-path of the quickest path is also optimal. The proposed algorithm runs in $O(rm + rn\log rn)$ time and uses at most $O(r(n + m))$ space, but it is shown that outperforms Martins and Santos (1997) algorithm (the most efficient until this date) in a small experiment presented in their paper.

Recently, Calvete et al. (2012) proposed an algorithm combining the simplicity of computing shortest paths with the explicit use in the algorithm of the transmission time and the value of σ. The time and space complexities match the complexities of the algorithm by Park et al. (2004). The authors claim that their algorithm performs well when comparing it with Chen and Chin (1990), Martins and Santos (1997) and Park et al. (2004) algorithms in an experiment considering a small number of different values of the arc capacities.

Pascoal et al. (2006) provide a survey on the quickest path problem. Several authors have also developed some extensions to the QPP problem. See for example (Chen, 1993; Chen and Hung, 1993; Lee and Papadopoulou, 1993; Chen, 1994; Chen and Hung, 1994; Xue, 1998; Xue et al., 1998; Kagaris et al., 1999; Calvete and del-Pozo, 2003; Lin, 2003; Calvete, 2004;Rao, 2004; Pascoal et al., 2005; Pascoal et al., 2007; Ruzika and Thiemann, 2012).

The contribution of this paper is a completely new algorithm based on the fact that the $s-t$ path with minimum transmission time is a *supported efficient solution* of the bicriteria path problem. This observation allow us to design a (ratio) labeling algorithm running in $O(rm + rn\log n)$ time, and using $O(n + m)$ space. The

* Corresponding author. Tel.: +34 922 845054; fax: +34 922 319202.
E-mail address: asedeno@ull.es (A. Sedeño-Noda).

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proposed algorithm avoids the two drawbacks pointed out by Park et al. (2004). First, the algorithm enumerates the supported efficient solutions of the bicriteria path problem until the optimal solution of the QPP is determined, taking into account the value of σ. Secondly, the algorithm works in a Dijkstra fashion, that is, only one application of a labeling method is performed, but this is not related to the optimality principle of the classical shortest path method.

In addition, Park et al. (2004) and Calvete et al. (2012) forgot a third drawback in their algorithms: the enlarging of the original network. Our algorithm does not need to enlarge implicitly or explicitly the original network. Moreover, we have kept the term $O(\log nr)$ (note that $O(\log n + \log r) = O(\log n)$ since $r \leqslant m < n^2$) in the time complexities of these algorithms to make clear the difference with our method, because in practice, this term has a computational time and space price.

Furthermore, we show that our method outperforms the previous methods in the different extensive experiments that we carried out. In particular, we have used the proposed algorithm to solve the QPP in USA road networks to show the robustness and scalability of the proposed algorithm. We note that only the algorithms of Martins and Santos (1997) and the one proposed in this paper are capable to solve the QPP in these networks. Any other algorithm enlarging the original network becomes impractical in a personal computer when it tries to solve the QPP on these instances.

The remaining sections of the paper are as follows: Section 2 introduces the QPP and describes some known results of the literature about of this problem. Section 3 contains the formulation of the QPP as a parametric programming problem and the study of its resolution. Section 4 proposes the ratio-labeling algorithm based on the parametric programming problem. This section provides the worst-case time and space complexity of this new algorithm. In Section 5, the comments on the computational experiment comparing the performance of the proposed algorithm and other known algorithms in the literature are given. Finally, Section 6 contains some additional comments and future lines of research.

2. The QPP and some previous results

Consider a directed graph $G = (V, A)$, where $V = \{1, \ldots, n\}$ is a set of n nodes and A is an arc set with m elements. For each arc $(i,j) \in A, c_{ij}$ is a nonnegative real number representing its lead time and u_{ij} is a nonnegative real representing the capacity of the arc. That is, u_{ij} represents the maximum number of items that can flow from node i to node j through arc (i,j) per time unit. The lead time c_{ij} is the time required for the items to traverse arc (i,j). Note that items arrive at node j within c_{ij} time units after being sent from node i. Therefore, the required transmission time of $\sigma > 0$ items through arc (i,j) is $c_{ij} + \sigma/u_{ij}$.

For all nodes, we denote by $\Gamma_i^- = \{j \in V | (j,i) \in A\}$ and by $\Gamma_i^+ = \{j \in V | (i,j) \in A\}$ the sets of predecessor and successor nodes, respectively.

The network has two different nodes from the rest: the *origin* node s and the *destination* node t. Let $i,j \in V$ be two nodes of $G = (V, A)$, so we define a directed path p_{ij} as a sequence $\langle i_1, (i_1, i_2), i_2, \ldots, i_{l-1}, (i_{l-1}, i_l), i_l \rangle$ of nodes and arcs satisfying $i_1 = i, i_l = j$ and for all $1 \leqslant w \leqslant l-1, (i_w, i_{w+1}) \in A$.

The lead time of a directed path p equals $C(p) = \sum_{(i,j) \in p} c_{ij}$ and the capacity of p is $U(p) = \min_{(i,j) \in p} \{u_{ij}\}$. Then, the transmission time required to send σ units of data through path p_{st} is defined as $T(p_{st}) = C(p_{st}) + \sigma/U(p_{st})$.

Assumption 1 (w.l.o.g.). *The network G contains a directed path from the origin node s to any node $i \in V - \{s\}$. When there is no path in G to some node i, then this node can be removed from G since it cannot lie on any $s - t$ path.*

Let P be the set of paths from s to t in G, then the QPP can be formulated as finding a $s - t$ path $p_{st}^* = \arg\min_{p_{st} \in P} T(p_{st})$. It is clear that there exists an optimal solution of the QPP which is loopless (without repeating nodes). Consider the next minsum–maxmin bicriteria path problem $\min_{p_{st} \in P}(C(p_{st}), 1/U(p_{st}))$, where the criteria $\max U(p_{st})$ appears as $\min 1/U(p_{st})$. The *objective space* is the image of the set of paths P under the previous objective functions.

Definition 1. A path $p \in P$ is called *efficient* if there does not exist any $p' \in P$ with $C(p') \leqslant C(p)$ and $U(p') \geqslant U(p)$ with at least one inequality being strict. The image $(C(p), 1/U(p))$ of p is called *non-dominated point.*

Definition 2. Supported efficient paths are those efficient paths that can be obtained as optimal paths of a weighted sum problem $\min_{p_{st} \in P}(\lambda_1 C(p_{st}) + \lambda_2/U(p_{st}))$ for some $\lambda_1 > 0$ and $\lambda_2 > 0$. All other efficient paths are *called non-supported.*

The *supported non-dominated points* lie on the lower-left boundary of the convex hull of the *objective space.*

Since the weighted sum problem with $\lambda_1 = 1 > 0$ and $\lambda_2 = \sigma > 0$ equals the QPP, we can rewrite the next result in Martins and Santos (1997) as:

Theorem 1. *Let $p_{st}^\sigma \in P$ be a quickest path for a given $\sigma \in \mathbb{R}^+$. Then, p_{st}^σ is a supported efficient path.*

This result was mentioned in Pelegrín and Fernández (1998), but it never was used to design a quickest path algorithm. Now, the question now is how to use this property. Martins and Santos (1997) give an initial answer. Let $P^E \subseteq P$ be the set of *supported* efficient paths and consider that this set is sorted such that $P^E = \{p_1, \ldots, p_r\}$ where $C(p_i) \leqslant C(p_{i+1})$ and $U(p_i) \leqslant U(p_{i+1})$, for any $i \in \{1, \ldots, r-1\}$. Without loss of generality, we will assume that $C(p_i) < C(p_{i+1})$ and $U(p_i) < U(p_{i+1})$, for any $i \in \{1, \ldots, r-1\}$ with $p_i \in P^E$.

In other words, we only need to compute all supported non-dominated points of the objective space (in the worst case). If two or more supported efficient paths have the same associated supported non-dominated point in the objective space, it is clear that it is only necessary to identify one of these paths in order to solve the QPP. The result in Martins and Santos (1997) is:

Theorem 2.

(1) p_1 *is a quickest path for* $\sigma \in \left(0, \frac{C(p_2)-C(p_1)}{U(p_2)-U(p_1)} \times U(p_2) \times U(p_1)\right]$

(2) p_i with $i \in \{2, \ldots, r-1\}$ *is a quickest path for*

$$\sigma \in \left[\frac{C(p_i) - C(p_{i-1})}{U(p_i) - U(p_{i-1})} \times U(p_i) \times U(p_{i-1}), \frac{C(p_{i+1}) - C(p_i)}{U(p_{i+1}) - U(p_i)} \times U(p_{i+1}) \times U(p_i)\right], \quad \text{for any } i \in \{2, \ldots, r-1\}.$$

(3) p_r *is a quickest path for* $\sigma \in \left[\frac{C(p_r)-C(p_{r-1})}{U(p_r)-U(p_{r-1})} \times U(p_r) \times U(p_{r-1}), +\infty\right]$.

Proof. (See Martins and Santos (1997)). □

The proof of this result given in Martins and Santos (1997) follows from elementary calculations to solve a system of inequalities. An alternative proof is based on the observation that the previous intervals correspond with the optimality intervals of the parametric programming problem $\min_{p_{st} \in P}(C(p_{st}) + \theta/U(p_{st}))$ with $\theta \geqslant 0$ (see Saaty and Gass, 1954). In our case, we need to solve the previous parametric programming problem starting with $\theta = 0$ until $\theta \leqslant \sigma$ in order to solve the QPP. However, the novel approach is that the second criterion is not linear in this parametric programming problem. Therefore, in the next section, we study the