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Discrete Optimization

When is rounding allowed in integer nonlinear optimization?

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ABSTRACT

In this paper we consider nonlinear integer optimization problems. Nonlinear integer programming has mainly been studied for special classes, such as convex and concave objective functions and polyhedral constraints. In this paper we follow an other approach which is not based on convexity or concavity. Studying geometric properties of the level sets and the feasible region, we identify cases in which an integer minimizer of a nonlinear program can be found by rounding (up or down) the coordinates of a solution to its continuous relaxation. We call this property *rounding property*. If it is satisfied, it enables us (for fixed dimension) to solve an integer programming problem in the same time complexity as its continuous relaxation. We also investigate the *strong rounding property* which allows rounding a solution to the continuous relaxation to the *next* integer solution and in turn yields that the integer version can be solved in the same time complexity as its continuous relaxation for arbitrary dimensions.

1. Integer nonlinear programming

It is well known that adding integrality constraints to continuous optimization problems increases their complexity in the majority of cases. A prominent example is linear programming, which can be solved in polynomial time by interior point methods, but becomes NP hard if integrality constraints are added. Integer linear programming has been widely studied and developed into a mature discipline. The theory developed in integer *nonlinear* programming is much less mature. According to Hemmecke, Köppe, Lee, and Weismantel (2010, chap. 15) who provide a recent overview, "integer nonlinear programming is still a very young field".

Integer nonlinear programming has been tackled by different communities. In the context of global optimization the main focus is to develop numerical procedures for solving nonlinear integer problems. Also techniques from integer linear programming are transferred to nonlinear integer programs. Results exist for integer concave minimization (equivalently integer convex maximization) which are based on the observation that a (quasi-)concave function attains its minimum (if it exists) at an extreme point of the feasible set. Hence an enumeration of all vertices of the convex hull of the integer points would solve the problem. More efficient structures based on total unimodularity for linear constraints allow polynomial procedures. Results for integer concave minimization can be found for example in De Loera, Hemmecke, Onn, and Weismantel (2008). The methods used for convex integer minimization are different; early approaches include the extension of branch and bound methods for linear integer programming (Gupta & Ravindran, 1985) which has been extended for example to convex quadratic integer programming (Buchheim, Caprara, & Lodi, 2012), or an extension of the cutting plane technique of Kelley (1960). More recently, outer approximation procedures have been suggested (Bonami et al., 2008). Research has also been done for special cases such as integer minimization of polynomial functions over polyhedral sets where an FPTAS is possible (De Loera, Hemmecke, Köppe, & Weismantel, 2006), of separable convex functions over polyhedral sets where proximity results are provided in Hochbaum and Shanthikumar (1990) or of strongly convex functions with Lipschitz continuous gradients over polytopes (Baes, Del Pia, Nesterov, Onn, & Weismantel, 2012). The idea of test sets for linear integer optimization problems (Graver, 1975) was transferred to some special cases of nonlinear integer optimization in Lee, Onn, and Weismantel (2008). Boolean nonlinear optimization has been considered for special types of problems, many of them motivated by discrete or network optimization as for example the quadratic assignment problem. An example for a recent approach can be found in Buchheim and Rinaldi (2007).

The approach we suggest in this paper is based on the *level sets* of the objective function: given a function $f : \mathbb{R}^n \to \mathbb{R}$ the *(sub-)level set* with respect to some level $z \in \mathbb{R}$ is defined as $\mathcal{L}_{\leqslant}(z) := \{x \in \mathbb{R}^n : f(x) \leqslant z\}$. Using level sets, the optimization problem $\min\{f(x) : x \in F\}$ for some function $f : \mathbb{R}^n \to \mathbb{R}$ and some set $F \subseteq \mathbb{R}^n$ can be reformulated as $\min\{z : \mathcal{L}_{\leqslant}(z) \cap F \neq \emptyset, z \in \mathbb{R}\}$, (where

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$\min\{\emptyset\} := \infty$), i.e., the goal is to identify the smallest level for which a feasible point exists. This approach is known as *graphical* approach in linear programming.

In this paper we treat the integrality constraint for an optimization problem $\min\{f(x) : x \in F, x \in \mathbb{Z}^n\}$ in the same way, i.e., we identify the smallest value $z \in \mathbb{R}$ for which $\mathcal{L}_\leqslant(z) \cap F \cap \mathbb{Z}^n \neq \emptyset$. This approach provides structural insight into the properties of an optimal solution. In particular we investigate the *rounding property*, i.e., in which cases one can solve the integer version of a nonlinear optimization problem by rounding (up or down) the coordinates of a solution to its continuous relaxation. If the rounding property holds for a problem, a finite dominating set is given by the set of all integer points adjacent to an optimal solution of its continuous relaxation. Therefore our results can be seen as proximity results (similar to Hochbaum & Shanthikumar (1990)) as we identify cases where $\|x^* - \bar{x}\|_\infty \leqslant 1$ for an integer solution x^* and a solution to the relaxation $\bar{x}$. It is known that convexity does not imply such a rounding property. However, we identify two different problem classes that satisfy the rounding property. These problems need not be convex and their objective functions not even continuous. Our analysis is done by investigating the geometric structure of the level sets.

The remainder of the paper is structured as follows. In the next section we introduce the rounding property. In Sections 3 and 4 we present two different geometric criteria which ensure the rounding property. Both lead to classes of nonlinear integer optimization problems which can be solved efficiently. The paper is ended by some conclusions and further research questions.

2. The rounding property

We consider integer nonlinear optimization problems given by some *objective function* $f : \mathbb{R}^n \to \mathbb{R}$ and some *feasible set* $F \subseteq \mathbb{R}^n$

$$(IP) \quad \min\{f(x) : x \in F, \ x \in \mathbb{Z}^n\}.$$

The *continuous relaxation* of (IP) is given by

$$(CP) \quad \min\{f(x) : x \in F\}.$$

Throughout the paper we assume that (IP) has an optimal solution x^* and that an optimal solution $\bar{x}$ to the continuous relaxation (CP) is known.

We investigate in which cases rounding the continuous minimizer $\bar{x}$ yield an optimal solution to (IP). By *rounding* we mean to round each coordinate of $\bar{x}$ up or down to the respective next integer, i.e., for $x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n$ we define

$$\mathrm{Round}(x) := \{y \in \mathbb{Z}^n : y_i \in \{\lfloor x_i \rfloor, \lceil x_i \rceil\} \ \forall i\}$$

as the set of integer points with rounded coordinates. For $x \in \mathbb{Z}^n$ we have $\mathrm{Round}(x) = \{x\}$ and for $x \in \mathbb{R}^n$ $\mathrm{Round}(x)$ contains at most 2^n points. We also investigate when rounding $\bar{x}$ to its *closest* integer point yields an optimal solution for (IP). We therefore define $\lfloor x \rceil$ to be the closest integer point to x, i.e., the coordinates x_i are rounded to the closest integer $y_i = \lfloor x_i \rceil$ for all $i = 1, \ldots, n$ using any fixed rule, e.g. the *round half up* rule in order to break ties.

Now we can introduce the following two rounding properties for nonlinear integer optimization problems given by some objective function f and a feasible set F.

Definition 2.1. We say that (f, F) has the *rounding property* if for any optimal solution $\bar{x}$ to (CP) there exists an optimal solution x^* to (IP), such that $x^* \in \mathrm{Round}(\bar{x})$.

If the rounding property holds, it guarantees that if (CP) is polynomially solvable then (IP) is also solvable in polynomial time for any fixed dimension, namely by first solving (CP) and then testing the at most 2^n points in $\mathrm{Round}(\bar{x}) \cap F$. This approach yields an efficient algorithm if the problem (CP) can be solved efficiently and the dimension n is rather small or if there are only a few points in $\mathrm{Round}(\bar{x}) \cap F$.

Note that the rounding property is trivially satisfied, but not at all helpful for the special case of Boolean optimization problems (BP) $\min\{f(x) : x \in F, \ x \in \{0,1\}^n\}$ for $f : [0,1]^n \to \mathbb{R}$, since enumerating the at most 2^n feasible 0/1 vectors is always an (inefficient) option to solve the problem.

In order to tackle also problems of type (BP), and to be more efficient when solving problems of type (IP), we introduce the *strong rounding property*.

Definition 2.2. We say that (f, F) has the *strong rounding property* if for any optimal solution $\bar{x}$ to (CP) there exists an optimal solution x^* to (IP), such that $x^* = \lfloor \bar{x} \rceil$.

The strong rounding property guarantees that if (CP) is polynomially solvable then (IP) is also solvable in polynomial time not only for fixed dimension.

Note that a function may have the (strong) rounding property without being continuous. However, continuity might be helpful in order to solve (CP). In the next two sections we derive two different classes of problems (specified by properties of f and F) satisfying the rounding property. To do so, the following reformulation of the (strong) rounding property is helpful.

Lemma 2.1.

(i) *(f, F) has the rounding property $\iff$ for any optimal solution $\bar{x}$ to (CP) and for all $x \in \mathbb{Z}^n \cap F$ we have that $\mathcal{L}_\leqslant(f(x)) \cap F \cap \mathrm{Round}(\bar{x}) \neq \emptyset$.*

(ii) *(f, F) has the strong rounding property $\iff$ for any optimal solution $\bar{x}$ to (CP) and for all $x \in \mathbb{Z}^n \cap F$ we have that $\lfloor \bar{x} \rceil \in \mathcal{L}_\leqslant(f(x)) \cap F$.*

Proof.

(i) "$\Rightarrow$" Let $\bar{x}$ be an optimal solution to (CP). Since (f, F) has the rounding property there exists $x^* \in \mathrm{Round}(\bar{x})$ optimal to (IP). This means that $x^* \in F$ and $f(x) \geqslant f(x^*)$ for any $x \in \mathbb{Z}^n \cap F$, hence $x^* \in \mathcal{L}_\leqslant(f(x))$ for all $x \in \mathbb{Z}^n \cap F$ and therefore $\mathcal{L}_\leqslant(f(x)) \cap F \cap \mathrm{Round}(\bar{x}) \neq \emptyset$.
"$\Leftarrow$" Let $\bar{x}$ be optimal for (CP). If $\mathcal{L}_\leqslant(f(x)) \cap F \cap \mathrm{Round}(\bar{x}) \neq \emptyset$ for any $x \in \mathbb{Z}^n \cap F$ there exists a $y \in \mathrm{Round}(\bar{x}) \cap F$ such that $f(y) \leqslant f(x)$ for any $x \in \mathbb{Z}^n \cap F$. This means one of the points in $\mathrm{Round}(\bar{x}) \cap F$ is optimal for (IP).

(ii) "$\Rightarrow$" If (f, F) has the strong rounding property, $x^* := \lfloor \bar{x} \rceil$ is an optimal solution to (IP) and hence contained in $\mathcal{L}_\leqslant(f(x))$ for all $x \in \mathbb{Z}^n \cap F$.
"$\Leftarrow$" Let $\bar{x}$ be optimal for (CP). If $\lfloor \bar{x} \rceil \in \mathcal{L}_\leqslant(f(x))$ for any $x \in Z^n \cap F$ we obtain that $f(\lfloor \bar{x} \rceil) \leqslant f(x)$ for any $x \in \mathbb{Z}^n \cap F$, hence $\lfloor \bar{x} \rceil$ is optimal for (IP). □

3. Cross-shaped level sets

To start with, we recall the definition of a star-shaped set: a set $M \subseteq \mathbb{R}^n$ is called *star-shaped* if a point $x_0 \in M$ exists, such that for any $y \in M$ the line segment $\lambda x_0 + (1 - \lambda) y, \lambda \in [0, 1]$, is contained in M. Boltyanski, Martini, and Soltan (1996) generalized this definition by introducing d-star-shaped sets for any norm $d : \mathbb{R}^n \to \mathbb{R}$ as follows: for $a, b \in \mathbb{R}^n$ denote by

$$[a, b]_d := \{x \in \mathbb{R}^n : d(a, x) + d(x, b) = d(a, b)\}$$

the *d-segment* of a and b with respect to the norm d. Then a set $M \subseteq \mathbb{R}^n$ is called *d-star-shaped* if a point $x_0 \in M$ exists, such that for any $y \in M$ the d-segment $[x_0, y]_d$ is contained in M.