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Polynomial cases of the economic lot sizing problem with cost discounts

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ABSTRACT

In this paper we study the economic lot sizing problem with cost discounts. In the economic lot sizing problem a facility faces known demands over a discrete finite horizon. At each period, the ordering cost function and the holding cost function are given and they can be different from period to period. There are no constraints on the quantity ordered in each period and backlogging is not allowed. The objective is to decide when and how much to order so as to minimize the total ordering and holding costs over the finite horizon without any shortages. We study two different cost discount functions. The modified all-unit discount cost function alternates increasing and flat sections, starting with a flat section that indicates a minimum charge for small quantities. While in general the economic lot sizing problem with modified all-unit discount cost function is known to be NP-hard, we assume that the cost functions do not vary from period to period and identify a polynomial case. Then we study the incremental discount cost function which is an increasing piecewise linear function with no flat sections. The efficiency of the solution algorithms follows from properties of the optimal solution. We computationally test the polynomial algorithms against the use of CPLEX.

1. Introduction

While the distribution of goods is usually driven by customer demand and is focused on service quality, the procurement of goods is a typically cost-driven phase (see Simchi-Levi, Kaminsky, & Simchi-Levi, 2007) in supply chain management. The procurement costs are mostly dependent on the cost of goods and on the transportation costs. One of the most frequently used transportation modes, especially when the shipment size is relatively small and the distance to be traveled is not too large, is less-than-truckload (LTL), where different shipments are consolidated in the same truck that delivers the goods to different customers. The assumption that the transportation cost charged by a carrier linearly depends on the shipped quantity usually makes the decision-making models simple and efficiently solvable. However, while this may be a reasonable assumption in a tactical planning phase, where the detailed cost structure is not essential, the linearity assumption is too simplistic in an operational phase. In fact, the tariffs applied by the carriers have a more complex structure. In LTL service, carriers encourage large shipments through non-linear cost structures. In most cases the tariffs have a piecewise linear structure. This is the case also in parcel rates and full-truck-load services (see Lapierre, Ruiz, & Soriano, 2004).

A company that optimizes its procurement phase has to take into account the procurement costs, that is the ordering costs and the inventory holding costs. Given the demand pattern, both kinds of costs are influenced by the ordering pattern. The classical model for the optimization of the procurement phase is the well known *single-item economic lot sizing problem* (ELS problem). A company has to satisfy a known demand over a discrete finite horizon. In each period the unit ordering and holding costs are known. Backlogging is not allowed and there is no constraint on the quantity ordered in each period. The problem consists in deciding when and how much to order in each period so that the total cost is minimized, where the total cost is the sum of the ordering cost and of the inventory holding cost (or simply inventory cost), both computed over the time horizon. The model, that is efficiently solvable, has been introduced by Wagner and Whitin (1958) and has attracted an enormous amount of research. The importance of the ELS problem in theory and practice has been widely recognized and the model appears in several textbooks on operations management (see, for example, Muckstadt & Sapra, 2009; Nahmias, 2001). The original paper has been recently reprinted in Management Science in Wagner and Whitin (2004) to evidence its relevance and impact on theory and practice. With respect to the basic Economic Order Quantity (EOQ) model due to Whitin (1957), in the ELS a

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multi-period problem is modeled, with the demand and the inventory holding costs possibly different in different time periods. The ordering cost, as in the EOQ model, is a fixed cost charged in each period where an order takes place. Several papers appeared in the literature that studied extensions of the original ELS problem. In most cases such extensions consider different ordering cost functions and we discuss the most interesting cases in the following. Also, efficient solution approaches to already known problems were presented (see, for example, Aggarwal & Park, 1993; Federgruen & Tzur, 1991; Wagelmans, van Hoesel, & Kolen, 1992). We refer to Brahimi, Dauzere-Peres, Najid, and Nordli (2006) for a survey on the ELS problem and to Robinson, Narayanan, and Sahin (2009), Buschkühl, Sahling, Helber, and Tempelmeier (2010), Hwang (2010), Hellion, Mangione, and Penz (2012) and Li, Ou, and Hsu (2012) for recent contributions to this problem.

As discussed by Nahmias (2001), there are three types of cost discount schemes commonly used in practice. The first type is the *all-units discount scheme* under which different discount rates apply to all units of different ordered quantities. The second type is the *incremental discount scheme* under which different discount rates apply to incremental ranges of the ordered quantity. The third type is the *Truck Load* (*TL*) *discount scheme* (also known as the carload discount scheme) under which the transportation service provider charges a LTL rate until the customer pays for the cost of a full truckload of Q units, which is typically less than the cost of shipping Q units at an LTL rate. Once the first truck is full, the customer again pays the LTL rate until the second truck is full, and so on. The TL discount scheme is also applicable to the situation where the customer is using a TL carrier and paying a TL rate for its TL shipments, although an LTL carrier is used for the delivery of the leftover quantities.

In this paper, we study the economic lot sizing problem with two different cost discount functions. The first is the *modified all-unit discount cost function* studied in Chan, Muriel, Shen and Simchi-Levi (2002) which alternates sections with positive slope and flat sections. The initial flat section indicates a minimum charge that must be paid for shipping a small quantity. This cost function corresponds to the case called in the industry *shipping a quantity, but declaring a lower quantity*. For instance, each pair of positive slope and flat sections can corresponds to the transportation capacity of one vehicle. If the shipped quantity is in the flat part, the cost of the full load vehicle is paid. Therefore, there is an incentive in sending quantities corresponding to full load vehicles. For the further practical motivations to study the modified all-unit discount cost function we refer to Chan, Muriel, Shen and Simchi-Levi (2002) and Chan, Muriel, Shen, Simchi-Levi and Teo (2002), where a variety of arguments is provided. It was shown in Chan, Muriel, Shen and Simchi-Levi (2002) that the problem is NP-hard. Moreover, the performance of practical policies was investigated in Chan, Muriel, Shen and Simchi-Levi (2002) for the single retailer case, whereas in Chan, Muriel, Shen, Simchi-Levi and Teo (2002) the results were extended to the multi-retailer case. In Shaw and Wagelmans (1998) the capacitated version of the problem was studied and a pseudo-polynomial algorithm was presented that can be applied to the uncapacitated case by setting the capacity to a large value.

In this paper, we investigate the problem where the cost function does not vary from period to period. A minimum charge is paid for small quantities. Then, an alternating sequence of sections with positive slope and flat sections follows. Both the sections with positive slope and the flat sections have identical length. The two lengths may be different. We will discuss in the paper the relevance of this assumption, that makes the cost function regular. We show that the problem with this piecewise linear cost function, referred to as *regular modified all-unit discount cost function*, can be solved in $O(I^2T^3)$ time, where I is the number of echelons, and T is the length of the discrete finite horizon. An echelon refers to two consecutive line segments, one with positive slope and one flat. The number of echelons is equal to the number of sections with positive slope.

The second cost function we consider is the *incremental discount cost function* which is an increasing piecewise linear function. We show that in this case the economic lot sizing problem can be solved with a more efficient polynomial algorithm, with complexity $O(T^2)$. The result relies on the property that it is never beneficial to order if the inventory available is sufficient to satisfy the current demand. This property does not hold in general for the economic lot sizing problem with *regular modified all-unit discount cost function*, but does hold if there is no minimum charge for small quantities.

To show how valuable the availability of polynomial algorithms is, we present computational results where we compare the solution time required by the algorithms with the solution of a mathematical programming formulation through CPLEX.

The structure of the paper is as follows. In Section 2 we define the problem we study in this paper and we give a mathematical formulation. In Section 3 we describe the problem with *regular modified all-unit discount cost function*, we prove properties of the optimal solution and present the polynomial algorithm for its solution. Then, in Section 4 we present properties of the optimal solution and a more efficient polynomial algorithm for the problem with *incremental discount cost function*. Finally, in Section 5 we present the computational results and in Section 6 we draw some conclusions.

2. The problem definition

In the economic lot sizing problem a facility faces known demands over a discrete finite horizon. At each period, the ordering cost function and the holding cost function are given and they can be different from period to period. There are no constraints on the quantity ordered in each period and backlogging is not allowed. The objective is to decide when and how much to order so as to minimize the total ordering and holding costs over the finite horizon without any shortages.

Let T be the length of the discrete planning horizon and denote by $t = 1, \ldots, T$ any period where an order may take place. If an order is placed at period t, then the delivery is assumed to be instantaneous. Let d_t be the known customer demand at period t, that we assume integer, and $h_t \geqslant 0$ be the unit inventory cost. We assume, without loss of generality, that the initial amount of inventory is $I_0 = 0$.

At period t the sequence of the operations is as follows. The inventory level is computed, then the product shipped at period t is received and, finally, the demand at period t is satisfied. Let $F_t(Q_t)$ be the ordering cost associated with an order of size Q_t placed at period t. The ordering cost function $F_t(Q_t)$ depends on the quantity through a cost discount function.

The first cost function we study is the *modified all-unit discount cost function* which is a piecewise linear function which alternates between increasing and flat sections, starting with a flat section. This cost function is studied in Chan, Muriel, Shen and Simchi-Levi (2002) and we now formally describe it. The function is depicted in Fig. 1. In Chan, Muriel, Shen and Simchi-Levi (2002) the use of this kind of function in practice is explained and motivated with a variety of arguments. The function is defined as follows. For small quantities of ordered product, that is quantities less than M_1, a minimum charge c is charged. Then, the function is structured in an infinite sequence of echelons, where each echelon i is defined by three integer values M_i, M'_i and M_{i+1}. The cost function is: