European Journal of Operational Research 237 (2014) 566–579

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

Asymmetric flow networks ☆

Norma Olaizola [a], Federico Valenciano [b,*]

[a] BRiDGE Group, Departamento de Fundamentos del Análisis Económico I, Universidad del País Vasco UPV/EHU, Avenida Lehendakari Aguirre 83, 48015 Bilbao, Spain[1]
[b] BRiDGE Group, Departamento de Economía Aplicada IV, Universidad del País Vasco UPV/EHU, Avenida Lehendakari Aguirre 83, 48015 Bilbao, Spain[1]

ARTICLE INFO

Article history:
Received 22 January 2013
Accepted 22 January 2014
Available online 30 January 2014

Keywords:
Networks
Non-cooperative network formation
Asymmetric flow
Stability
Dynamics

ABSTRACT

This paper provides a new model of network formation that bridges the gap between the two benchmark game-theoretic models by Bala and Goyal (2000a) – the one-way flow model, and the two-way flow model – and includes both as limiting cases. As in both the said models, a link can be initiated unilaterally by any player with any other in what we call an "asymmetric flow" network, and the flow through a link towards the player who supports it is perfect. Unlike those models, there is friction or decay in the opposite direction. When this decay is complete there is no flow and this corresponds to the one-way flow model. The limit case when the decay in the opposite direction (and asymmetry) disappears corresponds to the two-way flow model. We characterize stable and strictly stable architectures for the whole range of parameters of this "intermediate" and more general model. A study of the efficiency of these architectures shows that in general stability and efficiency do not go together. We also prove the convergence of Bala and Goyal's dynamic model in this context.

1. Introduction

The "area statement" corresponding to "Games, Information, and Networks" in the prestigious journal Operations Research states the following: "*A fundamental understanding of network behavior, including the nature of interconnections, issues of stability, and decision making place operations research methodology at the core of this emerging research program.*" This work seeks to contribute to the theoretical foundations of this research program at a methodological level.[2] More specifically, we address the issue of stability and dynamics in models of network formation. The possibility of generating complex structures from simple assumptions is interesting in various contexts.[3] In the context of networks, where complexity is often an unavoidable ingredient, it is also interesting to provide models that generate simple or relatively simple structures from simple assumptions. In this line of research, the seminal paper by Bala and Goyal (2000a) introduced two benchmark game-theoretic models of network formation: the "one-way flow" model, and the "two-way flow" model. In both models each player can unilaterally create (i.e. initiate and support) links at a given cost with any other player, and the objective is to receive the maximal flow at the minimum cost.[4] These two models differ in the way that the information and other benefits flow through the resulting network. In the one-way flow model the flow through a link runs only towards the player that supports it, while in the two-way flow model the flow through a link runs in both directions irrespective of who supports it. It is not surprising that such different models yield very different conclusions. In the absence of friction or decay, stable (in the sense of Nash equilibrium) networks are merely those minimally connected in either model, which means completely different

☆ We thank Coralio Ballester, Francesco Feri and José Félix Ramírez de Olano for helpful comments. Of course, the usual disclaimer applies. We gratefully acknowledge the important contribution of the Editor and two anonymous referees whose comments and suggestions allowed us to improve this paper considerably. This research is supported by the Spanish Ministerio de Ciencia e Innovación under projects ECO2012-31626 and ECO2012-31346. Both authors also benefit from funding from the Departamento de Educación, Política Lingüística y Cultura of the Basque Government for Grupos Consolidados IT869-13 and IT568-13.

[1] http://www.bridgebilbao.es.

* Corresponding author. Tel.: +34 94 6013696.
E-mail addresses: norma.olaizola@ehu.es (N. Olaizola), federico.valenciano@ehu.es (F. Valenciano).

[2] Increasing attention to networks in different contexts is perceptible in operational research literature. Usually, applied to different problems (e.g. flow optimization for evacuation planning (Lim, Zangeneh, Baharnemati, & Assavapokee, 2012), terrorism (Lindelauf, Hamers, & Husslage, 2013), network security (Skorin-Kapova, Furdeka, Aparicio Pardo, & Pavón Mariño, 2012)); but also addressing theoretical issues (e.g. models of opinion dynamics (Acemoğlu, Como, Fagnani, & Ozdaglar, 2013), network formation and stability (Dieker & Shin, 2013; Harmsen-van Hout, Herings, & Dellaert, 2013; Janssen & Monsuur, 2012; Monsuur, 2007).

[3] A remarkable example is Conway's Game of Life (Gardner, 1970).

[4] In Jackson and Wolinsky's (1996) model the formation of a link between two players requires the agreement of both. See Hellmann and Staudigl (2014) for a review on social network formation which includes a brief discussion of these basic game-theoretic models.

http://dx.doi.org/10.1016/j.ejor.2014.01.048
0377-2217/

architectures in each case.[5] The situation is also completely different in regard to stability in the *strict* Nash sense (i.e. where any unilateral deviation of any player means a loss): in the one-way flow model the "wheel" is the only strictly stable architecture, while in the two-way flow model the only strictly stable architecture is the "center-sponsored star". Despite the elegance of these results, the extreme simplicity of the emerging structures[6] – stars and wheels – is somewhat disappointing. Richer structures might have been expected to emerge even from simple assumptions.

Both models have been extended in several directions, but those extensions take one model or the other as their starting point and ignore the other.[7] To the best of our knowledge, no joint generalization has been provided so far, that is, no model has been proposed that integrates both models as particular cases. Notwithstanding, these two models look like extreme cases in an unspecified sense of an unspecified model, so different that they are hard to compare given the lack of intermediate models. One may, for instance, wonder about how the "transition" from wheels to stars occurs, but no transition is possible without intermediate situations or models. This paper provides such an "intermediate" model: a model of network formation that in fact includes Bala and Goyal's models as particular extreme cases.

As in those two models, a link can be initiated unilaterally by any player with any other in what we call an "asymmetric flow" network. As in the one-way flow model (without decay), the flow towards the player who initiates a link is perfect, without friction or decay, but *in the opposite direction it is not*. More precisely, if player i supports a link with j but player j does not support a link with i, all the information at node j reaches i without friction through this link, but *only a fraction* α of the information at node i reaches j through this link. Note that when $\alpha = 0$ this is the one-way flow model, while when $\alpha = 1$ the asymmetry of flow disappears and this is the two-way flow model. A second parameter of the model is the cost c (which we assume to be homogeneous across players) of initiating a link. Thus, Bala and Goyal's models are actually the limiting cases of this asymmetric flow model.

This new model has inherent interest per se, apart from the nice fact of including both benchmark models as extreme cases. It is worth noting that *asymmetry* actually occurs in different contexts. In many situations the flow of information between two agents is not equally fluent in both directions. Also, information which is supposed to flow only unidirectionally may suffer some sort of leakage in the opposite direction. Asymmetry also appears in telecommunications, where within the bandwidth of a system data speed and quantity differs from one direction to the other direction. In all these cases there is two-way asymmetric flow (or "asymmetric flow" for short).

Unilateral formation and asymmetric flow combine in some situations. For instance, the information flow within a network of informers, e.g., about catastrophes such as fires and earthquakes. In the academic world, quotations between authors and recommendations, where it is prestige that flows to some extent in both directions, are basically unilateral decisions with asymmetric consequences. Websites' links to other websites provide another example. This model can also be seen as complementary to the two-way asymmetric model of public relationships by Grunig (2001), where the point is to emit information and feedback is allowed but secondary. Finally, abundant asymmetric relationships are also conspicuous in the animal kingdom. A good example is provided by symbiotic relationships between individuals of different species from which two individuals benefit and which are often initiated unilaterally.

Thus, examples where asymmetry is present in bilateral relationships or transmissions and/or connections are created unilaterally abound. How each specific case should be treated is a different issue. In particular, the adequacy of a network formation model depends on the situation considered. In this respect, we have chosen basically the economically motivated model of network formation used by Bala and Goyal (2000a). On the plus side there is in the first place the simplicity of this decentralized formation model, which allows for a noncooperative game-theoretic analysis. This along with the asymmetric flow assumption provides a model which bridges the gap between Bala and Goyal's models and raises questions about stable, strictly stable and efficient architectures, the answers to which go beyond the dichotomy of "wheels versus stars". Here again wheels and center-sponsored stars are encountered (for certain configurations of values of the two parameters α and c), but so are new, richer structures such as root-oriented trees, with the oriented line as an extreme case among them, and other more complex architectures. We study the ranges for the parameters where such architectures are strictly stable, which of them overlap and where (in particular this may be the case for wheels and stars for certain ranges of these parameters). In fact all these structures, including the oriented wheel and the center-sponsored star, *turn out to be particular cases of a general architecture precisely described and characterized as the only one for which strict stability may hold*. A similar study about non-strict stability yields a characterization of the architecture of Nash networks. We also study the efficiency of these architectures. By contrast with Bala and Goyal's models, where efficiency and stability do not conflict, in the asymmetric flow model Nash and strict Nash networks may not be efficient and efficient networks may not be stable.

Finally, we address the extension of Bala and Goyal's dynamic model and results. This extension is achieved at the cost of a lengthy algorithmic constructive proof of the existence of a sequence of best responses which yields a strict Nash network starting from an arbitrary network.

The rest of the paper is organized as follows. In Section 2 an introductory example illustrates and anticipates in a simple context the general results presented later. Section 3 outlines the basic model and gives the necessary notation and terminology. Section 4 studies stability (Section 4.2) and strict stability (Section 4.1). Efficiency is briefly dealt with in Section 5. Section 6 applies Bala and Goyal's dynamic model to this setting. Finally, Section 7 summarizes the main conclusions and lines of further research.

2. An introductory example

A brief discussion of the conclusions of stability for the simplest case, a three-player society, provides an easy advance illustration of the general model and results presented and proved in subsequent sections. Consider a three-player society and assume that (i) each node has an information of value 1 for the other players; (ii) information flows through a link without loss in the direction of the player that supports it, but only a fraction α ($0 \leqslant \alpha \leqslant 1$) flows in the opposite direction; and (iii) the cost of initiating a link

[5] Connectedness in the one-way flow model requires the existence of an *oriented* path from any player to any other player, while in the two-way flow model it only requires the existence of a path, and the orientation of the links is indifferent. "Minimal" in both cases means that every link is necessary to keep the network connected. Thus, minimal connectedness yields different architectures in each model.

[6] In the presence of decay, things become much more complicated, but remain very different from one model to the other.

[7] The two-way flow model has received more attention, see e.g. Bala and Goyal (2000b), Goyal and Vega-Redondo (2005), Galeotti, Goyal, and Kamphorst (2006), McBride (2006), Feri (2007), Hojman and Szeidl (2008), and Bloch and Dutta (2009). For extensions of the one-way flow model see Galeotti (2006), Billand, Bravard, and Sarangi (2008), Derks, Kuipers, Tennekes, and Thuijsman (2009), and Derks and Tennekes (2009). We also address separately the extension of either model in the presence of constraints (Olaizola & Valenciano, 2013a, 2013b). Excellent books surveying this literature are Goyal (2007), Jackson (2008) and Vega-Redondo (2007). See also Jackson's (2010) survey.