European Journal of Operational Research 234 (2014) 583–596

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Invited Review

Evolution of social networks

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ARTICLE INFO

Article history:
Received 2 October 2012
Accepted 18 August 2013
Available online 30 August 2013

Keywords:
Evolution of networks
Game theory
Stochastic processes
Random graphs

ABSTRACT

Modeling the evolution of networks is central to our understanding of large communication systems, and more general, modern economic and social systems. The research on social and economic networks is truly interdisciplinary and the number of proposed models is huge. In this survey we discuss a small selection of modeling approaches, covering classical random graph models, and game-theoretic models to analyze the evolution of social networks. Based on these two basic modeling paradigms, we introduce co-evolutionary models of networks and play as a potential synthesis.

1. Introduction

The importance of network structure in social and economic systems is by now very well understood. In sociology and applied statistics the study of social ties among actors is a classical field, known as social network analysis (Wasserman & Faust, 1994). More recently a large scientific community, including game theorists, economists, as well as computer scientists and physicists, recognized the importance of network structure. In particular, the *dynamic evolution of networks* became an important question of research. Of course all these subjects put different emphasis on what is considered to be a "good" model of network formation. Traditionally, economists are used to interpreting observed network structure as equilibrium outcomes. Naturally, game theory is the predominant tool used in this literature. Computer scientists, on the other hand, prefer to think of network formation in terms of dynamic algorithms. Finally, physicists tend to think of networks as an outgrowth of complex system analysis, where the main interest is to understand and characterize the statistical regularities of large stochastic networks. Given this interdisciplinary character of the subject, the number of publications is enormous, and it is impossible to provide a concise survey covering the plethora of models developed in each of the above mentioned disciplines. For this reason, we have decided to focus in this survey on two, in our opinion, particular promising approaches to model the evolution of social and economic networks. We concentrate on dynamic models of network formation, using elements from random graph theory and game theory. These two approaches matured over the years, and some recent efforts have been made to combine them. This article summarizes a small body of these two streams of literature; it is our aim to convince the reader that random graph dynamics and (evolutionary) game theory have many elements in common, and we hope that this survey provides some ideas for future research on this young and interdisciplinary topic. However, before jumping into the details, let us give a short overview of topics which this survey covers, a pointer to the further relevant literature, and an acknowledgment of the literature which we shamefully exclude.

Section 2 starts with a short discussion of random graph models. These models are the basis for the statistical analysis of networks and have had a large impact on theoretical models of network evolution. Random graph models have a long tradition in social network analysis, and are the foundation of the recent literature on network evolution in computer science, mathematics and physics. Following the terminology of Chung et al. (2006), we focus on "off-line" models. Hence, we consider network formation models in which the number of nodes (the size of the population) is a given parameter.[1] The most general random graph model introduced in this survey belongs to the family of *inhomogeneous random graph models* (Bollobás, Janson, & Riordan, 2007). This class of random graphs is rather rich. It contains well-known *stochastic block-models* (Karrer & Newman, 2011), variants of the important *exponential random graph* (Snijders, Pattison, Robins, & Handcock, 2006), and the classical Bernoulli random graph model as special cases. Since these models are very well documented in the literature,

[1] "On-line" models are models in which the population is growing over time. This important class of models contains the very popular preferential attachment models (Barabási & Albert, 1999), which we are not touching in this survey. Excellent (technical) summaries of these models can be found in Newman (2003b) and, more rigorously, in Chung et al. (2006) and Durrett (2007).

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0377-2217/$ - see front matter
http://dx.doi.org/10.1016/j.ejor.2013.08.022

we are satisfied with a short summary of their main properties. The main purpose of this section is to introduce concepts, and to familiarize the reader with our notation. Nevertheless, we consider random graph models as the building block to synthesize random graph models with dynamic game theoretic ideas, and therefore we think it is useful to introduce them already at the beginning.

Section 3 presents an alternative approach of modeling networks, mainly developed by economic theorists. It uses game theoretical concepts to interpret network structures as *equilibrium phenomena* of strategically acting players who create and destroy links according to their own incentives. Our discussion is centered around two particularly important concepts: The semi-cooperative solution concept of *pairwise-stability* (Jackson & Wolinsky, 1996), and various modifications of *Nash equilibrium*. These concepts are static equilibrium notions, and the natural question which comes to one's mind is whether there are natural dynamic models supporting predictions based on such equilibrium concepts. This question leads us to consider an evolutionary approach to network formation. These models are used as an introduction to a particular interesting class of dynamic network formation models, called *co-evolutionary processes of networks and play* in Staudigl (2010). Section 4 is devoted to illustrate these types of models, which is our modest attempt to synthesize the strategic approach of network formation with a random graph approach. Section 5 summarizes the main points contained in this article and discusses some ideas for potential future research.

1.1. What this survey does not cover

Given the enormous number of network formation models, it was necessary to be selective in writing this survey. Hence, there are many important network models which we were not able to cover. Some notable omissions are the following; we restrict the discussion to network models where the *size* of the graph is fixed ("off-line" models). Networks with variable number of nodes ("on-line" models) are of course important, but they require different mathematical tools to be analyzed successfully. In particular, we believe that a different game theoretic approach would be needed to study such models.[2] Readers interested in the mechanics of growing networks should consult the book by Dorogovtsev and Mendes (2003), and the survey article by Albert and Barabási (2002). Another class of networks which we will not consider are *weighted graphs*. These models are very important for applications, and the physics community provides many interesting approaches to model such networks (see e.g. Barthélemy, Barrat, Pastor-Satorras, & Vespignani (2005) and Kumpula, Onnela, Saramäki, Kaski, & Kertész (2007), and the references therein). There are also some game–theoretic models on the formation of weighted networks, see e.g. Bloch and Dutta (2009). These are just preliminary studies, and we have the feeling that much more work on these kinds of networks will be needed before they should be included in a survey. Finally, we would like to point out that our main focus will be on evolutionary models of *undirected* networks. This does not mean that we think directed networks are less important. In fact, many real-world networks, such as traffic networks or the world-wide-web, are more naturally interpreted as directed graphs. However most of the game-theoretic concepts emphasize bilateral externalities, and thus admit a cleaner interpretation when links are undirected. Nevertheless, we provide some discussion of directed networks. In particular, the random graph models of Section 2 can be used to model the evolution of directed, as well as undirected networks, after straightforward modifications.

[2] For instance, a basic strategic decision in a network model with variable number of players is when and whether a single player should enter or leave the network. See Dutta, Ghosal, and Ray (2005) for a model in this direction. In evolutionary game theory, the standard model assumes a constant population size. However, it seems to be likely that allowing for population growth introduces for new phenomena absent from the stationary world. See Sandholm and Pauzner (1998) for an interesting study in this direction. A recent model of economic network formation in a non-stationary environment is Jackson and Rogers (2007).

1.2. Related literature

Most of the models which found no space in this article have been surveyed elsewhere. We just provide some cross references to the literature, and urge the reader to consult the references therein. Recent textbooks containing in-depth discussions on dynamic network formation are Chung et al. (2006) and Durrett (2007). These books focus on the mathematical aspects, and only mention rudimentary applications. With a slight bias towards economics applications, we recommend the beautiful books by Goyal (2007), Vega-Redondo (2007), Jackson (2008), and Easley and Kleinberg (2010). Additionally to textbook treatments, the reader may want to consult one of the numerous survey articles available. We just mention Jackson (2003), Newman (2003b), Jackson (2005), Van den Nouweland (2005), Goyal (2005), Goldenberg, Zheng, Fienberg, and Airoldi (2009). At the heart of our discussion is the evolution of networks, and we use the language of stochastic processes and/or game theory to formalize our ideas. Additionally we try to highlight the potential connections between these two seemingly separate modeling strategies. We hope that this integrated perspective on the evolution of networks makes this survey a good contribution to the literature and will inspire some people to work on this fascinating topic.

2. Stochastic models of network evolution

A statistical analysis of networks is usually based on elements of random graph theory. Random graph models are very flexible mathematical representations of interdependency relations. Indeed, the motivation for such classical models as the Markov graph model of Frank and Strauss (1986) was to develop a tractable representation of complicated interdependencies in empirical data. Random graph models in general provide a flexible framework to construct *statistical ensembles of networks* (i.e. probability spaces), which are parsimonious enough to get pointed predictions, and rich enough to be able to reproduce as many stylized facts the researcher is aiming to model. In this section our discussion is centered around a rather general class of a random graph process, which will be used as one pillar in our development of co-evolutionary processes of networks and play, starting in Section 4.

2.1. Random graphs

In this survey we use the terms "networks" and "graphs" interchangeably. A *graph* is a pair $G = ([N], E)$, where $[N] := \{1, 2, \dots, N\}$ is the set of *vertices* (or *nodes*), and $E \equiv E(G) \subset [N] \times [N]$ is the set of *edges* (or *links*). This notation applies to directed as well as undirected networks. If G is a directed graph, then $(i,j) \in E$ means that there is a directed edge from i to j. If G is undirected, then $(i,j) \in E(G)$ if and only if $(j,i) \in E(G)$. In such a symmetric setting it will be convenient to use the shortened notation ij for the link connecting node j and i. We denote the collection of graphs on N vertices by $\mathcal{G}[N]$. Elements of this set can either be directed or undirected networks, depending on the context.

A *random graph* is a probability space $(\mathcal{G}[N], 2^{\mathcal{G}[N]}, \mathsf{P})$. The probability measure $\mathsf{P} : 2^{\mathcal{G}[N]} \to [0, 1]$ assigns to each graph a weight, which should reflect the likelihood that a certain graph G is drawn from the set $\mathcal{G}[N]$, when performing a statistical experiment with distribution P. Thus, the underlying reference measure P it is chosen by the modeler. A historically very important random