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Continuous Optimization

Income drawdown option with minimum guarantee

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ABSTRACT

This paper deals with a constrained investment problem for a defined contribution (DC) pension fund where retirees are allowed to defer the purchase of the annuity at some future time after retirement.

This problem has already been treated in the unconstrained case in a number of papers. The aim of this work is to deal with the more realistic case when constraints on the investment strategies and on the state variable are present. Due to the difficulty of the task, we consider, as a first step, the basic model of Gerrard, Haberman and Vigna (2004), where interim consumption and annuitization time are fixed. We extend their model by adding a no short-selling constraint on the control variable and a final capital requirement constraint on the state variable. This implies, in particular, no ruin.

The mathematical problem is naturally formulated as a stochastic control problem with constraints on the control and the state variable, and is approached by the dynamic programming method. We write the non-linear Hamilton–Jacobi–Bellman equation for the problem and transform it into a dual one that is semi-linear, following a well-established duality procedure. In the special relevant case without running cost, we explicitly compute the value function for the problem and give the optimal strategy in feedback form. A numerical application ends the paper and shows the extent of applicability of the model to a DC pension fund in the decumulation phase.

1. Introduction

In countries where immediate annuitization is the only option available in defined contribution (DC) pension schemes, members who retire at a time of low bond yield rates have to accept a pension lower than the one available with higher bond yields (so-called annuity risk). In many countries, including Argentina, Australia, Brazil, Canada, Chile, Denmark, El Salvador, Japan, Peru, UK, US, the retiree is allowed to defer annuitization until some time after retirement, withdraw periodic income from the fund, and invest the rest of it in the period between retirement and annuitization. This allows the retiree to postpone the decision to purchase an annuity until a more propitious time. This flexibility is usually referred to as "income drawdown option" or "programmed withdrawal (option)".[1] For a detailed survey on the several forms of benefits provided by the programmed withdrawal option, we refer the interested reader to Antolin, Pugh, and Stewart (2008). There are often limits imposed on both the consumption and on how long the annuity purchase can be deferred. On the other hand, there is virtually unlimited freedom to invest the fund in a broad range of assets. While this option allows the retiree to aim for a final annuity higher than that purchasable at retirement, the evident drawback consists in the possibility of ruin, i.e. exhausting the fund while still alive. The three degrees of freedom of the retiree (amount of consumption, investment allocation, and time of annuitization), together with the important issue of ruin possibility, have been investigated in the actuarial and financial literature in many papers. Among others, we recall Albrecht and Maurer (2002), Blake, Cairns, and Dowd (2003), Di Giacinto and Vigna (2012), Gerrard, Haberman, and Vigna (2006), Gerrard, Højgaard, and Vigna (2012), Milevsky (2001), Milevsky, Moore, and Young (2006), Milevsky and Young (2007).

While the issue of ruin has been tackled in many papers, the problem of providing a minimum guarantee to the pensioner who takes the income drawdown option has not been considered in the literature, up to our knowledge. Nevertheless, the guarantee of a minimum level of ultimate annuity should be strong reason for taking programmed withdrawals. Moreover, the introduction of restrictions – which is more consistent with the financial

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[1] Other equivalent expressions are: phased withdrawal, scheduled withdrawal, allocated annuities, allocated pensions, allocated income streams.

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regulatory environment – makes possible a more accurate judgement on the effective tradeoff between the risks and the benefits provided by the income drawdown option. Motivated by these considerations, in this paper we fill in this gap in the literature, by defining and solving an optimal investment problem for the decumulation phase of a DC plan, where a minimum guarantee is provided and short-selling is forbidden.

The natural way of dealing with this problem is to formulate it as a stochastic optimal control problem with an appropriate choice of the state and control variables, of the constraints that they must satisfy, and of the optimizing criterion (utility or loss function). We choose the framework of Gerrard, Haberman, and Vigna (2004) and Gerrard et al. (2006, 2012) taking a quadratic target-based loss function. Indeed our approach could be used also to treat different objective functions. The three mentioned works consider similar models with an increasing number of degrees of freedom (i.e. control variables) but they do not solve the problems when constraints on the wealth and on the investment strategies are present.[2] Since the introduction of these constraints makes the problem very hard to attack and non treatable in the general case with the results of the known literature, we consider the simplest model (Gerrard et al., 2004), where the retiree is given only one degree of freedom, namely the investment allocation. The income withdrawn from the fund in the unit time is assumed to be fixed and the retiree is obliged to annuitize at a fixed future time T. In view of this fact, this paper must be considered as a first step towards a satisfactory treatment of the problem.

From the methodological point of view, following the approach of the papers mentioned above, we tackle the problem by the dynamic programming approach studying the associated Hamilton–Jacobi–Bellman (HJB) equation.

Notice that other methods can be used to deal with the same problem. In particular, in the case of optimal portfolio problems with capital constraints, both probabilistic duality methods and methods based on backward stochastic differential equations (BSDE) have been successfully employed in the literature. In this respect we observe that, among others, Basak and Shapiro (2001), Bielecki, Jin, Pliska, and Zhou (2005), El Karoui, Jeanblanc, and Lacoste (2005), Korn (1997), and Tepla (2001) are concerned with this kind of problem. More precisely, Basak and Shapiro (2001), Korn (1997), and Tepla (2001) deal with direct duality methods on the control problem – while we use duality at an analytic stage applying it to the HJB equation. However, differently from our case, in Korn (1997) the constraint is on the final average of the wealth, while in Basak and Shapiro (2001) there is a VaR-type constraint. The paper closest to ours seems to be Tepla (2001), where, as in our case, an almost sure constraint on the terminal wealth is imposed. What we get is from a purely analytic point of view the same result as in Tepla (2001).

Moreover, we should mention also the link of our problem with the rich class of mean–variance (MV) optimization problems in continuous-time. The well-known equivalence between MV-problems and expected utility maximization problems with quadratic utility function in the single-period framework can be extended to the continuous-time case (see for instance Bielecki et al., 2005; Korn, 1997; Vigna, in press; Zhou and Li, 2000). In the rich stream of literature on MV-optimization originating by the seminal paper Zhou and Li (2000), the work by Bielecki et al. (2005) solves a problem similar to ours, in a more general setting regarding the financial market. Their methodology is an extension of the risk neutral approach introduced by Pliska (1986) and boils down in presenting the optimal portfolio as the solution of a linear backward stochastic differential equation (BSDE). Related work can be found in Lim and Zhou (2002), who also use BSDEs to solve a mean–variance portfolio selection problem with random interest rate, appreciation rates and volatility coefficients. We mention also the paper Fu, Lari-Lavassani, and Li (2010) dealing with a MV portfolio selection problem with a borrowing constraint given by the presence of different interest rates for borrowing and lending.

If one chooses to deal with a dynamic programming approach – as we do – it turns out that the presence of a state constraint leads to proper boundary conditions for the HJB equation that prevent the possibility of finding simple explicit solutions to such an equation, unlike the papers mentioned above. Also a straight theoretical approach to the HJB equation – dealing, for instance, with a characterization of the value function as the unique viscosity solution and then with the proof of the regularity of viscosity solutions – is very problematic, since the HJB equation is a fully nonlinear, degenerate, non autonomous parabolic PDE. Therefore, we use a known procedure from portfolio optimization, which allows us to transform the original equation into a nicer looking dual one. This procedure has been used, e.g., in Elie and Touzi (2008), Gao (2008), Gerrard et al. (2012), Milevsky et al. (2006), Milevsky and Young (2007), Xiao, Zhai, and Qin (2007). In all such papers the dual equation is always linear.[3]

In our case the dual equation is in general semilinear and becomes linear when the current cost is zero. The general semilinear case is studied in the extended version of this paper (Di Giacinto, Federico, Gozzi, & Vigna, 2010) (to which the interested reader is referred) proving regularity of solutions to the dual equation and, consequently, to the original one. In this paper, for brevity, we set up the general model and then focus on the special and still significant case with no running cost, which allows, with a procedure similar to the above quoted papers, to find an explicit solution to the dual problem and come back with an explicit solution to the original one. So we can characterize the optimal strategy and wealth and perform a numerical simulation which allows to get some insights on the effects of the constraints on the optimal paths.

The availability of closed-form solutions to this problem is particularly important in the context of DC pension schemes. Indeed, using this model the retiree who takes the income drawdown option can decide about both the level of the minimum guarantee and that of a desired target. These levels are driven by the retiree's risk profile, the determination of which is typically an issue. Application of closed-form optimal policies coupled with numerical simulations for the risky asset provide the distribution of the annuity received upon ultimate annuitization, which helps the retiree to determine her own risk profile. All these features make this model – or possible evolutions of it – applicable by pension fund advisors in the decision-making process of retirees entering the decumulation phase of a DC scheme.

The remainder of the paper is organized as follows. In Section 2, we introduce the model and define the problem to be solved. Section 3 represents the theoretical core of the paper. Therein, we consider the problem, pass to the dual formulation, and show equivalence between the dual and the original problem. We then find closed-form solutions for the special case with no running cost. In Section 4, we show a numerical application that highlights the potential applicability to a DC pension plan. Section 5 concludes and outlines further research.

[2] In Gerrard et al. (2004), the only control variable is the investment strategy, in Gerrard et al. (2006) the control variables are the investment and the consumption policies, while in Gerrard et al. (2012) the retiree is allowed to choose the annuitization time, together with the investment-consumption policies.

[3] To this regard, it is worth stressing that in Schwartz and Tebaldi (2006) this procedure is applied in an incomplete market (due to the presence of an uninsurable income) giving rise to a dual equation which is still fully nonlinear. The authors approach this dual equation by means of a series expansion.