European Journal of Operational Research 234 (2014) 674–682

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Discrete Optimization

An aggressive reduction scheme for the simple plant location problem

Adam N. Letchford *, Sebastian J. Miller

Department of Management Science, Lancaster University Management School, Lancaster LA1 4YX, United Kingdom

ARTICLE INFO

Article history:
Received 26 November 2011
Accepted 6 October 2013
Available online 18 October 2013

Keywords:
Facility location
Combinatorial optimization
Integer programming

ABSTRACT

Pisinger et al. introduced the concept of 'aggressive reduction' for large-scale combinatorial optimization problems. The idea is to spend much time and effort in reducing the size of the instance, in the hope that the reduced instance will then be small enough to be solved by an exact algorithm.

We present an aggressive reduction scheme for the 'Simple Plant Location Problem', which is a classical problem arising in logistics. The scheme involves four different reduction rules, along with lower- and upper-bounding procedures. The scheme turns out to be particularly effective for instances in which the facilities and clients correspond to points on the Euclidean plane.

1. Introduction

The *Simple Plant Location Problem* (SPLP), sometimes called the *Uncapacitated Facility Location Problem* or *Uncapacitated Warehouse Location Problem*, is a fundamental and much-studied problem in the Operational Research literature. A formal definition is as follows. We are given a set I of facilities and a set J of clients. For any $i \in I$, the fixed cost of opening facility i is f_i. For any $i \in I$ and any $j \in J$, the cost of serving client j from facility i is c_{ij}. The goal is to decide which facilities to open, and to assign each client to an open facility, at minimum cost.

An excellent survey of the early work on the SPLP is given by Krarup and Pruzan (1983). In that survey, it is also formally proven that the SPLP is *NP*-hard, by reduction from the Set Covering Problem. More recent surveys on theory, algorithms and applications include Cornuéjols, Nemhauser, and Wolsey (1990), Labbé and Louveaux (1997), Labbé, Peeters, and Thisse (1995) and Verter (2011).

Some instances of the SPLP arising in practice have hundreds or even thousands of clients. Moreover, instances with large numbers of facilities and clients arise if one takes a *continuous* location problem and then 'discretises' it, by modeling continuous regions (approximately) as sets of discrete points. This led us, in our former paper (Letchford & Miller, 2012), to devise fast heuristics and bounding procedures for large-scale instances. In this paper, we move on to consider how to solve such instances to proven optimality (or near-optimality).

Pisinger, Rasmussen, and Sandvik (2007) introduced the concept of 'aggressive reduction' for large-scale combinatorial optimization problems. The idea is to spend much time and effort in reducing the size of the instance, using a suitable collection of variable-elimination tests. The hope is that the reduced instance will then be small enough to be solved by an exact algorithm.

In this paper, we present an aggressive reduction scheme for the SPLP, which uses four different reduction procedures. The scheme turns out to be particularly effective when the facilities and clients correspond to points on the Euclidean plane, and the cost of assigning a client to a facility is proportional to the distance between them. Indeed, for this case, we are able to solve instances that are significantly larger than those previously solved in the literature.

The structure of the paper is as follows. Section 2 is a brief literature review. In Section 3, we present two reduction procedures that are 'bound free', in the sense that no lower or upper bound is needed to apply them. In Section 4, we present some simple lower- and upper-bounding procedures, based on linear programming (LP). In Section 5, we present two more reduction procedures, that use the bounds in combination with LP duality. Extensive computational results are given in Section 6, and concluding remarks are given in Section 7.

We assume throughout the paper that the f_i and c_{ij} are positive integers. We also let m denote the number of facilities and n the number of clients.

2. Literature review

In this section, we review the main papers on relaxations, lower bounds, reduction rules and exact algorithms for the SPLP. There are also many papers on heuristics, meta-heuristics and approximation algorithms for the SPLP, but we do not cover them, for the sake of brevity. Instead, we refer the reader to the surveys mentioned in the introduction.

* Corresponding author. Tel.: +44 1524 594719.
E-mail address: A.N.Letchford@lancaster.ac.uk (A.N. Letchford).

0377-2217/$ - see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.ejor.2013.10.020

2.1. Linear programming relaxation

It is possible to formulate the SPLP as a 0–1 LP in several ways (see, e.g., Balinski, 1965; Cornuéjols, Nemhauser, & Wolsey, 1980; Efroymson & Ray, 1966; Krarup & Pruzan, 1983). The most commonly used formulation, due to Balinski (1965), is the following:

$$\min \sum_{i\in I} f_i y_i + \sum_{i\in I}\sum_{j\in J} c_{ij} x_{ij} \qquad (1)$$

$$\text{s.t.} \sum_{i\in I} x_{ij} = 1 \quad (\forall j \in J) \qquad (2)$$

$$y_i - x_{ij} \geqslant 0 \quad (\forall i \in I, \quad j \in J) \qquad (3)$$

$$x_{ij} \in \{0,1\} \quad (\forall i \in I, \quad j \in J) \qquad (4)$$

$$y_i \in \{0,1\} \quad (\forall i \in I). \qquad (5)$$

Here, x_{ij} indicates whether client j is assigned to facility i, and y_i indicates whether facility i is opened. The constraints (2) and (3) will be called *assignment constraints* and *variable upper bounds* (VUBs), respectively.

The LP relaxation is obtained by replacing the constraints (4), (5) with lower and upper bounds of 0 and 1, respectively. A key feature of this relaxation is that it typically gives a very good lower bound, and is often even integral (Ahn, Cooper, Cornuéjols, & Frieze, 1988; Efroymson & Ray, 1966; Morris, 1978; ReVelle, 1993). On the other hand, the presence of the VUBs makes the LP highly degenerate. Specialised primal simplex methods have been devised to cope with VUBs (Schrage, 1975; Todd, 1982), but they are not entirely satisfactory. For an alternative formulation of the SPLP as a set covering problem, see Cornuéjols et al. (1980).

2.2. Dual ascent and dual adjustment

In their seminal paper, Bilde and Krarup (1977) proposed to compute a lower bound by solving the dual of the LP approximately. The dual can be written, after some simplification, in the following form:

$$\max \sum_{j\in J} v_j \qquad (6)$$

$$\text{s.t.} \sum_{j\in J} w_{ij} \leqslant f_i \quad (\forall i \in I) \qquad (7)$$

$$v_j - w_{ij} \leqslant c_{ij} \quad (\forall i \in I, \quad j \in J) \qquad (8)$$

$$v_j \geqslant 0 \quad (\forall j \in J) \qquad (9)$$

$$w_{ij} \geqslant 0 \quad (\forall i \in I, \quad j \in J). \qquad (10)$$

Here, the v_j and w_{ij} are the dual variables for the assignment constraints and VUBs, respectively. Now, observe that there always exists an optimal solution to the dual in which

$$w_{ij} = \max\{0, v_j - c_{ij}\} \qquad (\forall i \in I, \quad j \in J). \qquad (11)$$

This leads to the following so-called *condensed* dual:

$$\max \sum_{j\in J} v_j \qquad (12)$$

$$\text{s.t.} \sum_{j\in J} \max\{0, v_j - c_{ij}\} \leqslant f_i \quad (\forall i \in I) \qquad (13)$$

$$v_j \geqslant 0 \quad (\forall j \in J).$$

Bilde and Krarup devised a fast heuristic, called *dual ascent*, for finding a good feasible solution to the condensed dual. The basic idea is to initialise the v_j at small values, and then repeatedly scan through the set of customers, increasing the dual values little by little until no more increase is possible.

In our own paper (Letchford & Miller, 2012), we showed that dual ascent runs in $\mathcal{O}(m^2 n)$ time. We described an improved version, which is faster in practice but has the same worst-case running time, along with a modified version which runs in only $\mathcal{O}(mn \log m)$ time, yet still produces reasonably good lower bounds.

Erlenkotter (1978) proposed an effective iterative method, called 'dual adjustment', for improving the dual solution generated by dual ascent. Several enhancements were also proposed by Körkel (1989). More recently, Hansen, Brimberg, Urosevic, and Mladenovic (2007) presented a variable neighborhood search (VNS) heuristic for the condensed dual. For the sake of brevity, we do not go into details.

2.3. Lagrangian relaxation

In Beasley (1993), Galvão and Raggi (1989), it was proposed to solve the dual approximately using Lagrangian relaxation, rather than dual ascent/adjustment. The assignment constraints (2) are relaxed, using a vector $\lambda \in \mathbb{R}^n$ of Lagrangian multipliers. The relaxed problem is then to minimize the Lagrangian

$$F(x, y, \lambda) = \sum_{i\in I} f_i y_i + \sum_{i\in I}\sum_{j\in J} (c_{ij} - \lambda_j) x_{ij} + \sum_{j\in J} \lambda_j,$$

subject to (3)–(5). This relaxation can be solved quickly, by computing for each $i \in I$ the 'Lagrangian reduced cost':

$$r_i = f_i - \sum_{j\in J} \max\{0, \lambda_j - c_{ij}\}, \qquad (14)$$

and then opening all facilities for which r_i is negative. The corresponding lower bound is:

$$\sum_{j\in J} \lambda_j - \sum_{i\in I} \max\{0, -r_i\}. \qquad (15)$$

The problem of finding optimal Lagrangian multipliers, the so-called *Lagrangian* dual, takes the form:

$$\max_{\lambda\in\mathbb{R}^n} \min_{(x,y)\in\{0,1\}^{mn+m}} F(x, y, \lambda).$$

This can be solved approximately using, for example, the subgradient method.

Very recently, Beltran-Royo, Vial, and Alonso-Ayuso (2012) applied to the SPLP a method called *semi-Lagrangian* relaxation. It gives tighter lower bounds, but at the cost of an increased running time.

2.4. Problem reduction

By 'problem reduction', we mean permanently fixing variables to 0 or 1, without losing any optimal solutions.

Körkel (1989) showed how to apply problem reduction to the SPLP, within a dual ascent or dual adjustment context. Let $\bar{v} \in \mathbb{R}^n$ denote a feasible solution to the condensed dual, let $\text{LB} = \sum_{j\in J} \bar{v}_j$ denote the corresponding lower bound, let UB be any upper bound, and, for all $i \in I$, define

$$s_i = f_i - \sum_{j\in J} \max\{0, \bar{v}_j - c_{ij}\}. \qquad (16)$$

Then, just like r_i in the previous section, s_i can be viewed as an estimate of the reduced cost of y_i in the primal. So, for any $i \in I$ such that s_i exceeds UB − LB, the variable y_i can be permanently fixed to 0, along with x_{ij} for all $j \in J$. Also, for any $i \in I$ and $j \in J$ such that

$$s_i + \max\{0, c_{ij} - \bar{v}_j\} > \text{UB} - \text{LB},$$

the variable x_{ij} can be permanently fixed to 0.

Beasley (1993) gave a slightly different problem reduction procedure, for use in a Lagrangian context. It uses the Lagrangian reduced costs r_i given in Eq. (14). Namely, if r_i is positive and LB + r_i > UB for any i, then y_i can be permanently fixed to 0 and, if