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Decision Support

Bayesian optimal knapsack procurement

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ABSTRACT

A budget-constrained buyer wants to purchase items from a shortlisted set. Items are differentiated by observable quality and sellers have private reserve prices for their items. The buyer's problem is to select a subset of maximal quality. Money does not enter the buyer's objective function, but only his constraints. Sellers quote prices strategically, inducing a knapsack game. We report the Bayesian optimal mechanism for the buyer's problem. We find that simultaneous take-it-or-leave-it offers are interim optimal.

1. Introduction

Consider a buyer who has a fixed budget to spend on items from a shortlisted set. The items differ in quality. Both, the qualities of the items and the buyer's budget are common knowledge. The quality of a subset of items is the sum of the individual qualities of its elements. A subset of higher quality is preferred to one of lower quality. Subsets of the same quality are considered as perfect substitutes. Each seller has private information about his reserve price for his item. The buyer's problem is to select a subset of items of maximal quality subject to his budget constraint.

Under complete information, the buyer faces a binary knapsack problem with qualities corresponding to values and reserve prices corresponding to weights in the standard notation. In the realm of incomplete information, any buying mechanism induces a game where sellers choose the weight of their item (i.e. the price they quote) strategically.

For an important application of this problem, consider government funds to subsidize R&D activities by private businesses. Typically, an agency has a fixed budget to spend to support research projects. Researchers apply for grants by submitting both a detailed plan of the research to be conducted and the associated cost. The quality of the proposals is then evaluated by a panel of independent experts. Based on these evaluations and on the stated cost the agency makes a funding offer. The agency's objective is to maximize the total quality of the supported projects.

What makes this problem non-standard is the procurer's objective, which requires special attention. In most other procurement problems money is part of the procurer's objective function in the sense that the procurer's welfare depends directly on the prices at which procurement happens. Not so here. Program managers do not value money in the sense that they assign a marginal value to it. Rather, they are supposed to support an additional project as long as they have enough money to do so. In other words, for them, there is no tradeoff between funding a shortlisted project and keeping the money. Hence, when we model this setting, money does only enter the procurer's constraints, not the objective function.

The contribution of this paper is to put on record an explicit solution for this procurement problem in which total transfers are bounded. Our problem falls into a new type of mechanism design problems in which there is a budget constraint on the sum of transfers (see the literature discussion below).

2. Literature

There is a long and fruitful tradition of combining game theory with operational research, (see Shubik, 2002) for a historical perspective. Many traditional OR problems have been analyzed within strategic settings, see the many examples given in Pardalos, Migdalas, and Pitsoulis (2008).

In particular, there is an active literature that analyzes traditional OR problems from a mechanism design perspective. Recent

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works have contributed to, e.g., inspection games (e.g. Avenhaus & Krieger, 2013; Fandel & Trockel, 2013), inventory management (e.g. Yan & Zhao, 2011), queuing (e.g. Guo & Hassin, 2011; Knight & Harper, 2013), advertising (e.g. Ashlagi, Monderer, & Tennenholtz, 2011; Nazerzadeh, Saberi, & Vohra, 2013), supply chain organization (e.g. Oliveira, Ruiz, & Conejo, 2013), machine scheduling (e.g. Heydenreich, Müller, & Uetz, 2010; Ashlagi, Dobzinski, & Lavi, 2012), market design (e.g. Muratore, 2011), auctions (e.g. Xia, Koehler, & Whinston, 2004, Dobzinski et al., 2010), and dynamic pricing (e.g. Gallien, 2006; Dasu & Tong, 2010; Pai & Vohra, in press).

Our problem is a game-theoretic variant of the knapsack problem (see e.g. Korte & Vygen, 2005). More precisely, consider the quality of the projects as their value and the required funding as their weight. Then the size of the knapsack is given by the budget and we have a knapsack problem where each item is controlled by a player who chooses the weight of its corresponding item strategically. Thus, our paper adds to the growing body of works that study the knapsack problem from a mechanism design perspective by adding incentive constraints to the original optimization problem, see, e.g., (Aggarwal & Hartline, 2006; Dizdar, Gershkov, & Moldovanu, 2011).[2]

A similar allocation problem, in the context of R&D subsidies, was first studied by Giebe, Grebe, and Wolfstetter (2006). They point out flaws in the widely applied rules for awarding R&D subsidies. Among several recommendations, they experimentally study the performance of open auctions as a means of inducing competition for funding. Ensthaler and Giebe (2013) provide a theoretical analysis of a modified open auction mechanism in a belief-free setting, as well as a description of the typical funding process and its flaws.

Despite its practical relevance, the problem has received surprisingly little attention in the mechanism design literature. To the best of our knowledge, there is no published paper on this issue. There are, however, two related unpublished manuscripts that study a mechanism design problem which is similar in nature to ours: In his Nancy L. Schwartz lecture, Maskin (2002) analyses the UK emissions reduction auction. In this auction, the UK government spent a predetermined fixed fund to pay firms to cut CO_2 emissions. Since firms' abatement costs are private information, this is a mechanism design problem. Maskin proceeds to derive the optimal ex post mechanism (that is, a mechanism which satisfies ex post IC, ex post IR and ex post budget balance) for special classes of distributions.

In an unpublished response, and independent of this work, Chung and Ely (2002) also analyze the interim problem and show that for every instance of the problem there exists a specification of the Baron–Myerson problem that has the same solution.[3] The results, where applicable, coincide with the ones presented in this paper. In particular, Chung and Ely also find that the optimal mechanism does not require ex post competition. In this paper we report the solution explicitly and in a constructive (algorithmic) fashion. Thus, our results can serve as a concrete benchmark when comparing/ranking different allocation mechanisms.

Recently, there has been interest in problems that involve the type of budget constraint as the one present in our problem in the computer science literature. Singer (2010) defines a whole new class of mechanism design problems that includes our problem as a special case. He calls mechanisms which have to satisfy a budget constraint as the one present in this problem *budget feasible mechanisms*. These problems are new because the budget constraint restricts the *payments* made by the mechanism in order to implement truthful reporting.[4] Also, while we restrict analysis to additive objectives, Singer (2010) covers a richer class of functions, such as subadditive functions. However, Singer as well as several recent articles that build on his work, restrict attention to a worst-case analysis and find mechanisms which are approximately optimal. In this sense, our paper complements their analysis.

3. The model

Assume that there are N potential sellers and let $i \in \{1, \ldots, N\}$ denote a typical seller. Each seller has an indivisible item to sell for which only he knows his private reserve price, θ_i, i.e., $\theta_i \in [\underline{\theta_i}, \overline{\theta_i}]$ with $0 \leqslant \underline{\theta_i} < \overline{\theta_i} < \infty$ for all i.

Denote $\theta := (\theta_1, \ldots, \theta_N)$ and $\Theta := [\underline{\theta}_1, \bar{\theta}_1] \times \cdots \times [\underline{\theta}_N, \bar{\theta}_N]$. As usual, the subscript $-i$ denotes a vector with the ith component removed (or a product with the ith factor removed).

We shall impose the classic assumption that there exist probability density functions $f_i : [\underline{\theta}_i, \bar{\theta}_i] \to \mathbb{R}$ for i's reservation price, θ_i, which is common knowledge. We assume that the f_i are continuous and strictly positive functions on $[\underline{\theta}_i, \bar{\theta}_i]$. Furthermore, let $F_i : [\underline{\theta}_i, \bar{\theta}_i] \to [0, 1]$ be the corresponding cumulative distribution functions such that

$$F_i(\theta_i) = \int_{\underline{\theta}_i}^{\theta_i} f_i(s_i) ds_i. \tag{1}$$

We also assume that the distributions of the θ_i are independent random variables. We denote the joint densities and cumulative distribution functions by f and F, respectively.

Items are differentiated by a fixed quality $w_i > 0$, which are common knowledge. We thereby assume that the differences in qualities can be expressed quantitatively, i.e. that the experts can state cardinal preferences for the individual projects.[5] We also assume that there is no statistical relationship between qualities and reserve prices that can be "exploited" by a mechanism.

Sellers are assumed to be risk-neutral. A seller with reserve price θ_i who receives a price t_i in return for selling his item with probability q_i has utility

$$u_i = t_i - \theta_i q_i. \tag{2}$$

Sellers are faced by a single buyer who has a finite budget $\mathcal{B} < \sum_{i=1}^{N} \bar{\theta}_i$ to acquire as many items as possible weighted by quality. In particular, money does not enter the buyer's objective function, it only enters his (budget) constraint. Thus, the buyer's problem is to find a mechanism that maximizes $E_\Theta\left(\sum_{i=1}^{N} w_i q_i(\theta)\right)$ subject to sellers' incentive and participation constraints and his own budget constraint.

More formally, let C_N denote the N-dimensional unit cube,

$$C_N = \{(x_1, \ldots, x_N) | 0 \leqslant x_i \leqslant 1\}. \tag{3}$$

Define a direct mechanism as a pair of functions q and t where

$$q : \Theta \to C_N, \tag{4}$$

$$t : \Theta \to \mathbb{R}^N. \tag{5}$$

Let $q_i(\theta)$ and $t_i(\theta)$ denote the ith components of q and t, respectively.

For now we shall only impose a *soft budget constraint*, (9), i.e. we will merely require that the budget is not exceeded in expectation. We will identify an optimal mechanism that satisfies the IR and IC

[2] In their papers, a profit maximizing seller wants to sell a given capacity to a set of buyers, i.e. the seller has a 'classic' objective function.

[3] See Baron and Myerson (1982).

[4] Another noteworthy result in this new strand of the literature is Ghosh et al. (2011) who derive an approximately optimal mechanism for purchasing sensitive information with a given budget in order to estimate a population statistic.

[5] In the case of R&D subsidies, practitioners (program managers) point out that their evaluation process gives them sufficient information to judge a proposal's quality. Typically, they award quality grades, like A, B, C, which can be used as part of a meaningful cardinal grading scheme, see Giebe et al. (2006).