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Decision Support

A new rule for source connection problems ☆

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ABSTRACT

In this paper we study situations where a group of agents require a service that can only be provided from a source, the so-called source connection problems. These problems contain the standard fixed tree, the classical minimum spanning tree and some other related problems such as the k-hop, the degree constrained and the generalized minimum spanning tree problems among others. Our goal is to divide the cost of a network among the agents. To this end, we introduce a rule which will be referred to as a painting rule because it can be interpreted by means of a story about painting. Some meaningful properties in this context and a characterization of the rule are provided.

1. Introduction

Situations in which a set of individuals should be connected to a source that provides a particular service are common in many fields, including telecommunications, water supply, etc. There are two approaches frequently associated with these problems. The first approach relates to how to build a network connecting all individuals to the source in order to fulfill a number of requirements that may be varied in nature, such as technical or economic. The second tackles the problem of how the total cost of the network should be divided between individuals. In this second approach, it seems reasonable that the distribution of costs among individuals who are connected to the source should be made somehow taking into account the network finally built. With this in mind, in the present paper we introduce and analyze a new rule for cost sharing for a wide range of source connection problems. In our model, in addition to considering the set of agents, the source and the cost matrix, we include a tree t, connecting all agents to the source. Let us draw attention to some well-known problems that fit into our general setting:

The classical minimum cost spanning tree problem. In this case t is a tree with minimal cost, computed for instance, with the algorithms of Kruskal (1956) or Prim (1957). Bird (1976), Granot and Huberman (1981), Bergantiños and Vidal-Puga (2007a), Sánchez-Soriano et al. (2010), among others, have studied the problem of how to divide the cost among the agents.

The generalized minimum spanning tree problem was introduced by Myung et al. (1995) where it is proved that it is *NP* hard. Given a graph whose nodes are partitioned into several clusters, it consists of finding a minimum cost tree with exactly one node from each cluster when the arcs are defined only between nodes which belong to different clusters. Pop et al. (2006) and Pop et al. (2011) present different procedures to find solutions within this class of problems. To the best of our knowledge, in this case how to share the cost of the tree among the agents has not been tackled up until the present time.

In the k-hop problem t is a tree where the number of arcs from the source to any agent cannot be more than a given constant k. In this class, the computation of a tree with minimal cost is *NP* hard, see Gouveia (1995) and Althaus et al. (2005). In fact, with $k = 2$ it includes a particular facility location problem when the facilities coincide with the agents, opening a facility at location of agent i is associated with the inclusion of arc $(0, i)$ in the tree (in this case agent i is assigned to that same facility and no additional cost is involved) and agent j is served by facility i when the arc (i, j) belongs to the tree. The problem of sharing the cost of the k-hop tree among the agents has only been studied in Bergantiños et al. (2012).

A degree-constrained spanning tree is a spanning tree where the maximum node degree is limited to a certain number k. In this case, t is a tree where the number of adjacent arcs to any node

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cannot be more than k. While the unconstrained minimum spanning tree problem can be solved in polynomial time, this problem is *NP* hard, cf. Garey and Johnson (1979). In Narula and Ho (1980) three procedures to construct such a tree are proposed. As far as we know, the problem of sharing the cost of the tree among the agents has not yet been studied.

Standard fixed tree networks are studied by Granot et al. (1996). In this type of problems the unique tree is t. Granot et al. (1996), Maschler et al. (2010), Koster et al. (2001), and among others, give attention to the problem of sharing the cost in these situations.

In all the above-mentioned problems, a tree to be constructed is finally selected. In our model we assume that the tree to be constructed is selected among those satisfying certain conditions or constraints, but without modeling how to do it. The tree constructed could be the cheapest one, but not necessarily (the problem could be *NP* hard). Therefore our model lies somewhere in between the fixed tree model (when alternative trees are not taken into account) and other models like the minimum cost spanning tree problem where the selected tree depends directly on the cost matrix and it is always the cheapest one.

In Bergantiños et al. (2012) a rule for dividing the cost of a tree among the agents in the k-hop minimum cost spanning tree is proposed. This rule, even it satisfies some nice properties, does not depend on the structure of the tree constructed, only on the total cost of the tree. It seems reasonable that the way of distributing the cost of building this tree, should be related to the structure of the tree itself. In this paper we propose a rule for dividing the cost of a tree among the agents in a general setting. Our rule is based on a painting story, like some other rules for fixed tree problems.

Besides, we present an axiomatic characterization of the rule using the next properties: isolated agents, cost monotonicity, equal treatment inside (C,t)-components and cone-wise additivity. An agent is isolated when he is very far (in terms of cost) from the other agents in such a way that there are not savings if this agent connects to the source through the others and viceversa. Isolated agents must pay his connection cost to the source. Cost monotonicity says that, if the tree to be constructed does not change, thus the rule should be monotonic on the cost matrix C. Equal treatment inside (C,t)-components says that agents that are connected through a path of cost 0 in t, must pay the same. Cone-wise additivity says that the rule should be additive in cones.

The cost monotonicity property has some implications to be highlighted. This property could be considered as a solidarity property because it says that if some cost increases or decreases, all agents should be affected in the same way (nobody should pay less when the cost increases and nobody should pay more when the cost decreases). Besides, it implies that the cost shares will only depend on the tree finally built. This fact makes the solution easier to compute but has some drawbacks, for example, the no consideration of the cost of the links outside the tree constructed means that we ignore much of the information and hence we lose interesting information about the negotiation power or threats of agents involved. Then, this property is more adequate when there is a central authority deciding the cost shares of the agents than when the agents should decide the cost shares through a bargaining procedure among themselves.

Finally, we think that our approach is suitable in a broad context, which includes as particular cases, the generalized minimum cost spanning tree problem, the k-hop problem and the degree-constrained spanning tree problem.

The rest of the paper is organized as follows. In Section 2 we recall some basic notions for the source connection problems and introduce our model. Section 3 is devoted to the introduction of the new rule. In Section 4 we present some meaningful properties and provide a characterization of our rule, which coincides with the folk rule in the classical minimum cost spanning tree problem. Thus, a new characterization of the folk rule is obtained, together with an easy way of calculating it. Section 5 concludes.

2. Source connection problems

In this section we introduce the model we are interested in and the notation used throughout the paper.

Let $\mathcal{N} = \{1, 2, \ldots\}$ be the set of all possible agents (nodes). Consider a set $N \subset \mathcal{N}$ of agents who are interested to be connected to a *source*, denoted by 0, in order to obtain a certain service. Let $N_0 = N \cup \{0\}$ and $N = \{1, \ldots, |N|\}$, where $|N|$ denotes the number of elements of the set N.

A *network* g over N_0 is a subset of $\{(i,j) : i,j \in N_0\}$ and its elements are called *arcs*. Given a network g and a pair of different nodes i and j, a *path* from i to j(in g) is a sequence of different arcs $\{(i_{s-1}, i_s)\}_{s=1}^{p}$ that satisfy $(i_{s-1}, i_s) \in g$ for all $s \in \{1, 2, \ldots, p\}, i = i_0$ and $j = i_p$. It is said that $i,j \in N$ are *connected* (in g) if there is a path from i to j. $S \subset N_0$ is connected (in g) if each pair of agents $i,j \in S$ are connected in $g_S = \{(i,j) \in g : i,j \in S\}$.

A *tree*, t is a network where for each pair of nodes $i,j \in N_0$ there is a unique path in t from i to j. We denote the set of all networks over N_0 as $\mathcal{G}^N$ and the set of all trees as $\mathcal{T}^N$. Given a tree t, the *predecessor set* of a node i in t is $Pre(i,t) = \{j \in N_0 : j$ is in the unique path from i to the source$\}$. We assume that $i \notin Pre(i,t)$ and $0 \in Pre(i,t)$ when $i \neq 0$. For notational convenience, $Pre(0,t) = \emptyset$. The *immediate predecessor* of node i in t, denoted as i^0, is the node that comes immediately before i in the unique path from 0 to i, that is, $Pre(i,t) = \{i^0\} \cup Pre(i^0, t)$. Following this definition, we usually write $t = \{(i^0, i)\}_{i \in N}$.

A *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ in N_0 represents the cost of each arc between any pair of nodes. We assume that $c_{ij} = c_{ji} \geqslant 0$ for all $i,j \in N_0$ and that $c_{ii} = 0$ for all $i \in N_0$. We denote the set of all cost matrices over N_0 as $\mathcal{C}^N$. Given $C, C' \in \mathcal{C}^N$ we say that $C \leqslant C'$ if $c_{ij} \leqslant c'_{ij}$ for all $i,j \in N_0$.

A cost matrix C is *simple* if $c_{ij} \in \{0, x\}$ with $x \in \mathbb{R}_+$ for all $i,j \in N_0$. From Theorem 1 in Norde et al. (2004), for each cost matrix C there is a family $\{C^r\}_{r=1}^{\alpha}$ of simple matrices satisfying three conditions:

1. $C = \sum_{r=1}^{\alpha} C^r$.
2. For each $r \in \{1, \ldots, \alpha\}$ there is a network g^r such that $c^r_{ij} = x^r$ if $(i,j) \in g^r$ and $c^r_{ij} = 0$ otherwise.
3. There is $\sigma : \{(i,j)\}_{i,j \in N_0, i<j} \to \{1, 2, \ldots, \frac{|N|(|N|+1)}{2}\}$ such that if $i,j,k,l \in N$ with $i<j, k<l$, and $\sigma(i,j) \leqslant \sigma(k,l)$, then $c_{ij} \leqslant c_{kl}$ and $c^r_{ij} \leqslant c^r_{kl}$ for all $r \in \{1, \ldots, \alpha\}$.

Given a pair (N_0, C) and $g \in \mathcal{G}^N$, we define the *cost* associated with g as

$$c(N_0, C, g) = \sum_{(i,j) \in g} c_{ij}.$$

When there is no ambiguity, we will write $c(g)$ or $c(C,g)$ instead of $c(N_0, C, g)$.

Our goal is to connect all agents to the source, directly or indirectly, through a tree which fulfills certain conditions. Let T denote the set of all trees in $\mathcal{T}^N$ with these conditions. The objective is, first, to select a tree t in T and, then, to divide the cost among the agents. We should select a tree t in T with minimal cost. Depending on the set T this tree t could be exactly computed or approximated (cf. Gupta and Könemann (2011) for a survey on approximation algorithms for network design).

Formally a *source connection problem*, *scp*, is a triple (N_0, C, t), where N is the set of agents, 0 is the source, C is the cost matrix and $t \in T$ is the tree connecting all agents to the source to be constructed.