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Stochastically weighted stochastic dominance concepts with an application in capital budgeting

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ABSTRACT

The problem of comparing random vectors arises in many applications. We propose three new concepts of *stochastically weighted dominance* for comparing random vectors X and Y. The main idea is to use a random vector V to scalarize X and Y as V^TX and V^TY, and subsequently use available concepts from stochastic dominance and stochastic optimization for comparison. For the case where the distributions of X, Y and V have finite support, we give (mixed-integer) linear inequalities that can be used for random vector comparison as well as for modeling of optimization problems where one of the random vectors depends on decisions to be optimized. Some advantages of the proposed new concepts are illustrated with the help of a capital budgeting example.

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1. Introduction

Many decision situations require comparison of random vectors. Examples of such situations arise in multi-period reward in dynamic programming (Dentcheva and Ruszczyński, 2008), multi-criteria decision making (Hu and Mehrotra, 2012), risk adjusted budget allocation (Hu et al., 2011), health applications (Armbruster and Luedtke, 2010), and capital budgeting Graves et al. (2003). The concept of stochastic dominance can be used for comparing random variables and vectors (see, e.g., Shaked and Shanthikumar, 1994; Müller and Stoyan, 2002 and Levy, 2006 for comprehensive treatments of the topic). A well-known approach to compare random variables is to compare their expected utility, for a given utility function u that represents the decision maker's preferences. Stochastic dominance circumvents the problem of assuming knowledge of a decision maker's utility function by requiring that $\mathbb{E}[u(X)] \geqslant \mathbb{E}[u(Y)]$ for *all* u in a certain class $\mathfrak{U}$ of utility functions. The concept of utility-based comparison naturally generalizes to the multivariate case by using multivariate utilities. However, as observed by Zaras and Martel (1994) and Nowak (2004), it may become conservative in that setting. Moreover, testing dominance relationship in the multivariate case can be difficult, although recent work by Armbruster and Luedtke (2010) provides some new tools for that.

In this paper we propose three new concepts of *stochastically weighted dominance*. These concepts build on the idea of weighted scalarization of the random vectors. We informally present the basic ideas below to facilitate discussion on their benefits. We say that X dominates Y in the multivariate linear sense if

$$\mathbb{E}[u(v^TX)] \geqslant \mathbb{E}[u(v^TY)], \quad \forall v \in \mathfrak{V}, \ \forall u \in \mathfrak{U}. \tag{1.1}$$

A standard approach, known as multivariate linear stochastic dominance, is to use $\mathfrak{V} = \mathbb{R}^n_+$ (see, e.g., Müller and Stoyan, 2002 and Dentcheva and Ruszczyński, 2009). Homem-de-Mello and Mehrotra (2009) and Hu et al. (2012) allow $\mathfrak{V}$ to be an arbitrary polyhedral and a convex set, respectively. In our new concepts we will allow a stochastically weighted scalarization by introducing a probability measure indicating the relative importance of each vector of weights. The use of random weights was also studied in Hu and Mehrotra (2012) to develop stochastic-weight robust models for multi-stochastic objective optimization problems without the framework of stochastic dominance.

In the first concept, called stochastically weighted dominance in average (see definition in (SWD-Avg)), we require that condition (1.1) hold not for all $v \in \mathfrak{V}$ but just on the average with respect to a distribution supported on the set $\mathfrak{V}$. In the second concept, called stochastically weighted dominance with chance (see definition in (SWD-Chance)), we require that condition (1.1) hold for a given fraction of the $v \in \mathfrak{V}$ instead of all v. The third concept, called almost stochastically weighted dominance (see definition in (SWD-Almost)), can be roughly interpreted as allowing a tolerance on the right-hand side of (1.1).

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The proposed new concepts provide additional modeling flexibility and potential computational advantages over current techniques. For example, they allow for the use of random discount rates in net present value type calculations in capital budgeting where future cash flow and predicted discount rates are random, thus resulting in a more accurate representation of future uncertainty. Moreover, the new concepts provide a systematic approach to reconcile diverse stakeholder opinions elicited by a decision maker by allowing the use of a distribution-based scalarization in optimization problems with multiple risk constraints. This situation arises, for example, when the weights are collected from a large survey on a public issue (see e.g., Hu et al., 2011).

In addition, the stochastic weighted dominance concepts introduced here provide a way to relax condition (1.1), which in certain situations may be too strict as it requires that dominance hold for every $v \in \mathfrak{V}$. Finally, the concepts in (SWD-Avg) and (SWD-Almost) lead to linear inequalities when the second order dominance is used to compare the scalarized vector. In comparison, known approaches for the model in (1.1) require solving an optimization problem involving concave minimization problems (see Homem-de-Mello and Mehrotra, 2009 and Hu et al., 2012), which are difficult to solve.

The remainder of this paper is organized as follows. In Section 2 we review some concepts of stochastic dominance, including the notion of multivariate linear dominance under a weight region, and provide some basic results. In Section 3 we formally describe new concepts of stochastically weighted dominance and provide their equivalent utility representations. In Section 4 we present linear and mixed-integer linear systems of inequalities corresponding to the new dominance relationships. This is done in the context where one of the random vectors is an outcome depending on decision variables in an optimization problem. In Section 5 we present an example from capital budgeting to highlight the salient features of our proposed concepts. The appendices contain the proofs of the theorems presented in the paper, as well as a simple example that illustrates the effect of the various notions of dominance on the size of the feasibility set of an optimization problem.

2. Review of multivariate stochastic dominance

We start by reviewing the concept of nth order stochastic dominance (see, e.g., Dentcheva and Ruszczyński, 2003). Consider an underlying probability space $(\Omega, \mathcal{F}, P)$. For $m \geqslant 1$, let $\mathcal{L}_n^m$ denote the set of m-dimensional random vectors X defined on $(\Omega, \mathcal{F}, P)$ satisfying $\mathbb{E}[\|X\|_\infty^n] < \infty$. We will omit the index m when $m = 1$.

We write the cumulative distribution function (cdf) of a real-valued random variable ξ as

$$F_1(\xi;\eta) := P(\xi \leqslant \eta).$$

Furthermore, for $\xi \in \mathcal{L}_{n-1}$ where $n \geqslant 2$, define recursively the functions

$$F_j(\xi;\eta) := \int_{-\infty}^{\eta} F_{j-1}(\xi;t)\,dt, \quad j = 2,\ldots,n.$$

It is useful to note the equivalence given by Proposition 1 in Ogryczak and Ruszczyński (2001)

$$F_j(\xi;\eta) = \frac{1}{(j-1)!}\mathbb{E}\left[\left((\eta-\xi)_+\right)^{j-1}\right], \quad j = 2,\ldots,n. \tag{2.1}$$

It implies that, for any n, the condition $\xi \in \mathcal{L}_{n-1}$ suffices to ensure that $F_n(\xi;\eta) < \infty$ for all η.

A random variable $\xi \in \mathcal{L}_{n-1}$ is said to stochastically dominate another random variable $\zeta \in \mathcal{L}_{n-1}$ in nth order if

$$F_n(\xi;\eta) \leqslant F_n(\zeta;\eta), \quad \forall \eta \in \mathbb{R}. \tag{2.2}$$

We use the notation $\xi \unrhd_{(n)} \zeta$ to indicate this relationship.

At this point it is useful to establish the utility based interpretation of the first and second order stochastic dominance. It is well known that

$$\xi \unrhd_{(n)} \zeta \iff \mathbb{E}[u(\xi)] \geqslant \mathbb{E}[u(\zeta)], \quad \forall u \in \mathfrak{U}_n, \tag{2.3}$$

whenever the expectations exist. In the above statement, $\mathfrak{U}_1$ is the set of all nondecreasing functions $u : \mathbb{R} \mapsto \mathbb{R}$ and $\mathfrak{U}_2$ is the set of all nondecreasing concave functions $u : \mathbb{R} \mapsto \mathbb{R}$. We refer to Müller and Stoyan (2002),Dentcheva and Ruszczyński (2003) and Rachev et al. (2008) for a more detailed discussion of stochastic dominance. We shall avoid the repetition of the clause "whenever the expectations exist", and always understand (2.3) in that context.

The concept of nth order dominance can be readily extended to the multivariate linear case as defined below.

Definition 2.1. A random vector $X \in \mathcal{L}_{n-1}^m$ dominates another random vector $Y \in \mathcal{L}_{n-1}^m$ in the linear nth order with respect to set $\mathfrak{V} \subseteq \mathbb{R}_+^m$ if

$$F_n(v^T X;\eta) \leqslant F_n(v^T Y;\eta) \quad \forall \eta \in \mathbb{R},\ \forall v \in \mathfrak{V}. \tag{RSD}$$

Since the above inequality must hold for all $v \in \mathfrak{V}$, we shall sometimes refer to this relationship as "robust weighted" stochastic dominance.

Theorem 2.2 below shows that without loss of generality we can assume that $\mathfrak{V} \subseteq \Delta := \{v \in \mathbb{R}^m : \|v\|_1 \leqslant 1\}$. This property generalizes the results in Homem-de-Mello and Mehrotra (2009) who consider the case of the second order dominance.

Theorem 2.2. *Let $\mathfrak{V}$ be a non-empty convex set, and denote $\widetilde{\mathfrak{V}} := \mathrm{cone}(\mathfrak{V}) \cap \Delta$ (cone denotes the conical hull). Then,*

(i) $v^T X \unrhd_{(1)} v^T Y$ for all $v \in \mathfrak{V}$ if and only if $v^T X \unrhd_{(1)} v^T Y$ for all $v \in \widetilde{\mathfrak{V}}$.
(ii) For $n \geqslant 2$, $v^T X \unrhd_{(n)} v^T Y$ for all $v \in \mathfrak{V}$ if and only if $v^T X \unrhd_{(n)} v^T Y$ for all $v \in \mathrm{cl}(\widetilde{\mathfrak{V}})$ (cl) denotes the closure of a set).

Proof. See Appendix B.1. □

3. Stochastically weighted dominance

In this section we introduce notions of stochastically weighted stochastic dominance. We shall use the notation V instead of v to emphasize that the weights are random. We assume that the m-dimensional random vector V and the random-outcome vectors X and Y are defined on the same probability space. Such an assumption is made only for notational convenience and does not lead to loss of generality, since we can always construct a random vector V' on the same probability space as X and Y having the same distribution as V. Note also that V may in principle be correlated with X and Y, which allows for incorporating information on X and Y to influence the choice of weights. Let us define the conditional distribution function

$$F_1(V^T X;\eta|V) := P(V^T X \leqslant \eta|V),$$

and denote

$$F_j(V^T X;\eta|V) := \int_{-\infty}^{\eta} F_{j-1}(V^T X;t|V)\,dt, \quad j = 2,\ldots,n.$$

We note that $F_j(V^T X;\eta|V)$ is a nonnegative random variable. Moreover, from (2.1) we have that

$$F_j(V^T X;\eta|V) = \frac{1}{(j-1)!}\mathbb{E}\left[\left((\eta - V^T X)_+\right)^{j-1}|V\right], \quad \forall \eta \in \mathbb{R} \quad \text{a.s.}. \tag{3.1}$$

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