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Decision Support

Preference inference with general additive value models and holistic pair-wise statements

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ABSTRACT

Additive multi-attribute value models and additive utility models with discrete outcome sets are widely applied in both descriptive and normative decision analysis. Their non-parametric application allows preference inference by analyzing sets of general additive value functions compatible with the observed or elicited holistic pair-wise preference statements. In this paper, we provide necessary and sufficient conditions for the preference inference based on a single preference statement, and sufficient conditions for the inference based on multiple preference statements. In our computational experiments all inferences could be made with these conditions. Moreover, our analysis suggests that the non-parametric analyses of general additive value models are unlikely to be useful by themselves for decision support in contexts where the decision maker preferences are elicited in the form of holistic pair-wise statements.

1. Introduction

We consider a decision problem of (partially) ordering a set of alternatives that are deterministically evaluated in terms of $n > 1$ attributes. Preferences of the Decision Maker (DM) are assumed to be representable by an additive multi-attribute value function that is indirectly defined through holistic pair-wise judgments (i.e. alternative a is weakly preferred over alternative b, $a \succsim b$). Note that although we consider, for compatibility with the existing literature, only multi-attribute value models, the results also apply directly to multi-attribute utility models with discrete outcome sets. The Robust Ordinal Regression (ROR) methodology (Corrente, Greco, Kadziński, & Słowiński, in press; Corrente, Greco, & Słowiński, 2013; Greco, Mousseau, & Słowiński, 2008; Greco, Kadziński, Mousseau, & Słowiński, 2011, 2012; Greco, Kadziński, & Słowiński, 2011; Kadziński, Greco, & Słowiński, 2012a, Kadziński, Greco, & Słowiński, 2012b, 2012c, 2013a) enables non-parametric analyses of sets of preference models compatible with the given holistic pair-wise judgments. ROR methods supply the DM with two kinds of results: possible and necessary preference relations that express whether an alternative is weakly preferred over another one with some or all compatible preference models, respectively. ROR has been implemented initially for the non-parametric analyses of additive value models in UTA^{GMS} (Greco et al., 2008).

The necessary and possible relations are computed in UTA^{GMS} by solving Linear Programs (LPs). However, their computation time can be too high for practical purposes, especially in decision contexts where the problem needs to be solved repeatedly, or with larger problem sizes. Also, it has not been known what exactly can be inferred through non-parametric analyses of additive models when the DMs express preferences as holistic pair-wise statements. Answering this question is relevant for the economical sciences as a whole, because many regression models implicitly assume an axiomatic foundation in terms of value theory (also known as utility theory with riskless decisions), and as the LP based approach seems to be appropriate also for descriptive decision analysis (Graf, Vetschera, & Zhang, 2013).

In this paper, we prove necessary and sufficient conditions for single preference statement inference, and sufficient conditions for multiple statement inference (Section 2). We report results of our computational experiments that measured the amount of different types of preference inferences as well as inferences that could not be made using our propositions (Section 3). The paper ends with a discussion of the propositions and the results.

2. Analysis of the general additive value model

We consider a multi-attribute decision problem where a finite set of alternatives M is evaluated on a set of attributes indexed with $S = \{1, \ldots, n\}$. We denote the evaluation of alternative $A \in M$ on attribute i with A_i. Without loss of generality, we assume the evaluations to be cardinal and the alternatives Pareto-optimal.

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The DM preferences over M are representable with a value function $u : M \to \mathbb{R}$,

$$u(A) \geqslant u(B) \iff A \succsim B. \tag{1}$$

We assume mutual preferential independence of the DM's preferences (Keeney & Raiffa, 1976) and therefore u is additive and composed of partial value functions u_i, that are, without loss of generality, assumed to be monotonically increasing,

$$u(A) = \sum_{i \in S} u_i(A_i). \tag{2}$$

The set of all such value functions is $\mathscr{U}$. Let P be the set of weak preference ($\succsim$) statements provided by the DM. A value function u is *compatible to* P if, $\forall (A \succsim B) \in P$, $u(A) \geqslant u(B)$. The set of all value functions compatible to P is denoted by $\mathscr{U}^P \subseteq \mathscr{U}$. P is said to be *non-conflicting* if $\mathscr{U}^P \neq \emptyset$. In what follows, we assume P to be non-conflicting.

Definition 1. For $A, B \in M$ such that $(A \succsim B) \notin P$, if $u(A) \geqslant u(B), \forall u \in \mathscr{U}^P$, we say we are able to *infer* $A \succsim B$ using P.

Note that if $u(A) \geqslant u(B), \forall u \in \mathscr{U}^P$, then in the UTA^{GMS} terminology A is necessarily preferred to B.

The conditions we derive for preference inference are based on examining the partial value function domains whose corresponding ranges are constrained in size by the preference statements:

Definition 2. P_i^+ and P_i^- are, $\forall i \in S$,

$$P_i^- = \bigcup_{\substack{(X \succsim Y) \in P \\ Y_i > X_i}} [X_i, Y_i] \tag{3}$$

$$P_i^+ = \bigcup_{\substack{(X \succsim Y) \in P \\ X_i > Y_i}} [Y_i, X_i] \tag{4}$$

Note that when $|P| = 1, P_i^+ = [Y_i, X_i]$ if $Y_i < X_i$ and $P_i^+ = \emptyset$ otherwise, and $P_i^- = [X_i, Y_i]$ if $X_i < Y_i$ and $P_i^- = \emptyset$ otherwise. Using P_i^- and P_i^+, the following lemma provides a condition under which preference inference is impossible.

Lemma 1. *For $A, B \in M$, if $\exists k \in S : B_k > A_k$ and $[A_k, B_k] \not\subseteq P_k^-$, then $\exists u \in \mathscr{U}^P : u(B) > u(A)$.*

Proof. Consider an arbitrary $u \in \mathscr{U}^P$. We construct a new value function u' by modifying u so that $u' \in \mathscr{U}^P$ and $u'(B) > u'(A)$.

Let $u^{sum} = \sum_i [u_i(\max_{X \in M}\{X_i\}) - u_i(\min_{X \in M}\{X_i\})]$. Furthermore, let $D = [A_k, B_k] \setminus P_k^-$ and let D^* be an arbitrary non-empty convex subset (an interval) of D, i.e. D^* is an interval fully contained in D. Let $u_i' = u_i, \forall i \in S \setminus \{k\}$, and define $u_k'(x)$ for an $\varepsilon > 0$ as

$$u_k'(x) = \begin{cases} u_k(x) & \text{if } x \leqslant \min\{D^*\}; \\ u_k(x) + u^{sum} + \varepsilon & \text{otherwise.} \end{cases} \tag{5}$$

Observe that u' differs from u only by the partial value function u_k'. Moreover, for each non-empty convex subset $[x, y] \subseteq P_k^-, u_k'(x) - u_k'(y) = u_k(x) - u_k(y)$, and for $[y, x] \subseteq P_k^+, u_k'(x) - u_k'(y) \geqslant u_k(x) - u_k(y)$. Therefore, $\forall (X \succsim Y) \in P$,

$$u'(X) - u'(Y) = \sum_{i \in S \setminus \{k\}} [u_i'(X_i) - u_i'(Y_i)] + u_k'(X_k) - u_k'(Y_k) \tag{6}$$

$$= \sum_{i \in S \setminus \{k\}} [u_i(X_i) - u_i(Y_i)] + u_k'(X_k) - u_k'(Y_k) \tag{7}$$

$$\geqslant \sum_{i \in S} [u_i(X_i) - u_i(Y_i)] \tag{8}$$

$$\geqslant 0 \Rightarrow u' \in \mathscr{U}^P. \tag{9}$$

Furthermore,

$$u'(B) - u'(A) = \sum_{i \in S \setminus \{k\}} [u_i'(B_i) - u_i'(A_i)] + u_k'(B_k) - u_k'(A_k) \tag{10}$$

$$= \sum_{i \in S \setminus \{k\}} [u_i(B_i) - u_i(A_i)] + u_k(B_k) + u^{sum} + \varepsilon - u_k(A_k) \tag{11}$$

$$= u^{sum} - \sum_{i \in S} [u_i(A_i) - u_i(B_i)] + \varepsilon \tag{12}$$

$$\geqslant \varepsilon, \tag{13}$$

and $\varepsilon > 0 \Rightarrow u'(B) > u'(A)$. □

Lemma 1 provides a simple condition that can be used for checking whether $\exists u \in \mathscr{U}^P : u(B) > u(A)$, or in UTA^{GMS} terminology, whether A is not necessarily preferred to B. This is illustrated in Fig. 1.

Corollary 1. *For $A, B \in M$, if $u(A) \geqslant u(B), \forall u \in \mathscr{U}^P$, then $[A_i, B_i] \subseteq P_i^-, \forall i \in S : A_i < B_i$.*

Proof. Assume that $u(A) \geqslant u(B), \forall u \in \mathscr{U}^P$, and that $\exists k \in S : A_k < B_k$ and $[A_k, B_k] \not\subseteq P_k^-$. Now Lemma 1 implies that $\exists u \in \mathscr{U}^P : u(B) > u(A)$, which contradicts the assumption. □

Next we provide another condition under which preference inference is impossible. It considers a single preference statement only.

Lemma 2. *For $A, B, X, Y \in M$ and $P = \{(X \succsim Y)\}$, if $\exists k \in S : A_k \geqslant B_k$ and $P_k^+ \not\subseteq [B_k, A_k]$, then $\exists u \in \mathscr{U}^P : u(B) > u(A)$.*

Proof. If $\exists t \in S : [A_t, B_t] \not\subseteq P_t^-$, then $B_t > A_t$, and by Lemma 1 $\exists u \in \mathscr{U}^P : u(B) > u(A)$. In the remainder of the proof we consider the other case in which $[A_i, B_i] \subseteq P_i^-, \forall i \in S$. In particular, because all alternatives in M are Pareto optimal, $\exists \ell \in S : B_\ell > A_\ell$ and $[A_\ell, B_\ell] \subseteq P_\ell^-$, i.e. $Y_\ell \geqslant B_\ell > A_\ell \geqslant X_\ell$. To complete the proof, we construct a value function u and show that $u(X) \geqslant u(Y)$ and $u(B) > u(A)$.

For all $i \in S \backslash \{k, \ell\}$, let $u_i(x) = 0$. Furthermore, for an $\varepsilon > 0$, define u_ℓ as

$$u_\ell(x) = \begin{cases} \varepsilon, & \text{if } x > A_\ell; \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

If $X_k > A_k > Y_k$, define u_k as

$$u_k(x) = \begin{cases} 2\varepsilon, & \text{if } x > A_k; \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

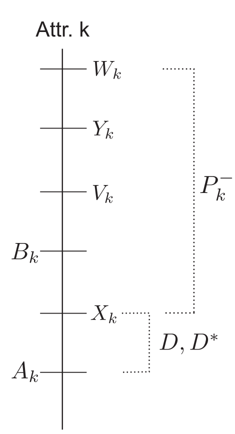

Fig. 1. An example problem with two preference statements, $P = \{(X \succsim Y), (V \succsim W)\}$, in which the condition of Lemma 1 holds. As the interval $[A_k, B_k]$ is not completely within P_k^-, the value interval $[u_k(\min\{D^*\}), u_k(\max\{D^*\})]$ can be increased by an arbitrary large amount, and therefore $\exists u \in \mathscr{U}^P : u(B) > u(A)$.