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Decision Support

New results on the relationship among risk aversion, prudence and temperance

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ABSTRACT

This note studies the relationships between different aspects of agent's preferences toward risk. We show that, under the assumptions of non-satiation and bounded marginal utility, prudence implies risk aversion (imprudence implies risk loving) and that temperance implies prudence (intemperance implies imprudence). The implications of these results for comparing risks in the cases of increase in risk, increase in downside risk and increase in outer risk are discussed.

1. Introduction

A wide literature in the theory of decision making under uncertainty focuses on agents' attitude towards risk by analyzing features of the utility function. Some of the different concepts identified have great relevance to economic problems. Three of the most important are risk aversion, prudence and temperance.

Risk aversion was originally introduced by Pratt (1964), and describes the idea that agents dislike risks. Risk aversion is relevant in a wide range of economic issues, including insurance, portfolio problems and asset pricing (for a survey see, for instance, Gollier, 2001). When we assume that preference are represented by the utility function $U(x)$ where x is the agent's consumption, the agent's attitude toward risk aversion is related to the second derivative of this function. In particular, an agent is risk averse if $\frac{d^2U(x)}{dx^2} = U''(x) < 0$ while she is risk lover if $\frac{d^2U(x)}{dx^2} = U''(x) > 0$.

Prudence was first introduced in the literature on precautionary saving (see Kimball, 1990; Leland, 1968; Sandmo, 1970) as the desire to increase saving in the presence of income risk. Beyond its relevance for saving decisions, prudence has also proved to be important in other fields, such as optimal prevention (Eeckhoudt & Gollier, 2005; Menegatti, 2009) and the agent's preferences about skewness (Chiu, 2005; Menezes, Geiss, & Tressler, 1980). In utility theory, an agent is prudent if $\frac{d^3U(x)}{dx^3} = U'''(x) > 0$ while she is imprudent if $\frac{d^3U(x)}{dx^3} = U'''(x) < 0$.

Temperance, the third concept, was introduced by Kimball (1992) in the study of the effect of labor income risk on the fraction of saving devoted to risky investment, and is defined as "moderation in accepting [independent] risks" (Kimball, 1992, p. 160). This feature of preferences has been shown to be relevant for optimal decisions in the presence of background risk (Eeckhoudt, Gollier, & Schlesinger, 1996; Gollier & Pratt, 1996), in the study of the agent's preferences when risks differ in kurtosis (Menezes & Wang, 2005) and for the analysis of the effects on saving of an increase in the downside risk of future labor income (Eeckhoudt & Schlesinger, 2008). In utility theory, an agent is temperate if $\frac{d^4U(x)}{dx^4} = U''''(x) < 0$ while she is intemperate if $\frac{d^4U(x)}{dx^4} = U''''(x) > 0$.

Risk aversion, prudence and temperance are concepts which concern different aspects of the agent's attitude toward risk, and which are related to three derivatives of different orders of the utility function. However, since they are related to different features of the same preferences (i.e. different derivatives of the same function), it is important to understand if there are relationships between them. As will be shown in the literature overview below, few results have been derived on this issue.

The aim of this work is to provide new insight into this field. We will show in particular that the different aspects of the agent's attitude toward risk are not independent and that, under plausible assumptions on the utility function, prudence implies risk aversion (imprudence implies risk loving) and temperance implies prudence (intemperance implies imprudence). Moreover, since risk aversion, prudence and temperance are related to agents' choices in many economic problems, the existence of these relationships is potentially relevant for many issues. This work analyses some of the most relevant of these issues, by studying comparison between different risks and by providing a simple application to a precautionary saving problem.

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The work proceeds as follows. Section 2 provides a short literature overview. Section 3 introduces the assumptions and studies the relationships between temperance, prudence and risk aversion. Section 4 presents the implications of the results derived for increase in risk, increase in downside risk and increase in outer risk. Section 5 provides a simple application to precautionary saving. Section 6 concludes.

2. Literature overview

There are two main strands of literature relevant for the present work.

This work studies the relationships between different features of preferences toward risk, represented by the signs of derivatives of different order of the utility function. Despite its potentially important implications, this issue is not largely studied in utility theory.[1] To my knowledge the only explicit finding in this field was made by Menegatti (2001) who proved that if an agent is non-satiated and risk averse and if the third derivative of her utility function is assumed to be invariant in sign then this sign must be positive and the agent must be prudent.

It is useful to emphasize that this result is different and is in fact complementary to results presented here. Menegatti (2001) showed sufficient conditions (non-satiation and invariant sign of the third derivative of the utility function) which ensure that risk aversion implies prudence. The present work shows sufficient conditions ensuring that prudence implies risk aversion (as well as conditions ensuring that temperance implies prudence). Note that the sufficient conditions in the two works are not the same.

A further issue examined in the same strand of literature is the relationship between the index of absolute risk aversion ($-U''(x)/U'(x)$) and the index of absolute prudence ($-U'''(x)/U''(x)$). Some results have been published in this field. First Kimball (1992) proved that if preferences exhibit decreasing absolute prudence (i.e. $-U'''(x)/U''(x)$ is decreasing in x) for every $x > 0$ then they also exhibit decreasing absolute risk aversion (i.e. $-U''(x)/U'(x)$ is decreasing in x) for every $x > 0$. Maggi, Magnani, and Menegatti (2006) extended this result to two other cases: the relationship between increasing absolute prudence and increasing absolute risk aversion and the relationship between constant absolute prudence and constant absolute risk aversion. Furthermore they show that, in general, the number of local minima (maxima) of the index of absolute risk aversion is not larger than the number of local minima (maxima) of the index of absolute prudence. Finally, in a related field, Eeckhoudt and Schlesinger (1994) showed conditions ensuring that if one agent has a higher index of absolute prudence than another then she also has a higher index of absolute risk aversion and vice versa.

As stated in the introduction, this work presents implications and potential applications of its findings on risk aversion, prudence and temperance for comparing risks. The literature studying comparison of risks is thus also relevant for the present paper. The first seminal contribution was by Rothschild and Stiglitz (1970) who introduced the concept of increase in risk as a case where a risk can be obtained from another risk by a mean-preserving spread, which leaves the mean of the distribution unchanged and increases the variance.[2] In the same field, Menezes et al. (1980) define an increase in downside risk as a case where a risk can be obtained from another risk by a mean–variance–preserving contraction, which leaves mean and variance of the distribution unchanged and increases negative skewness.[3] Finally Menezes and Wang (2005) study an increase in outer risk as a case where a risk can be obtained from another risk by a sequence of outer transformations, which leaves mean, variance and skewness of the distribution unchanged and raises kurtosis.[4]

These comparisons are relevant to many areas, and in general to all problems where an agent chooses between different risky prospects. A standard case is when a firm compares different investment projects where each of them has different possible returns with different probabilities to occur. Similarly, risk is compared in finance whenever an agent compares different portfolios characterized by different random returns with different distributions. Such problems have been widely studied in the literature. For a summary of results, see Gollier (2001) and Levy (2006).

Other applications of different kinds of risk changes have recently been presented in various fields. Eeckhoudt and Schlesinger (2008) studied conditions ensuring that different kinds of changes in risk increase precautionary saving. A simple application in this field is presented at the end of this work too. Eeckhoudt, Schlesinger, and Tsetlin (2009), Section 4.3 analyzed the effect of risk changes on production localization. Chiu and Eeckhoudt (2010) applied comparison of risks to the choice of agent's labor supply. Finally Baiardi and Menegatti (2011) studied the effects of risk changes of different kinds on environmental policy.[5]

3. Risk aversion, prudence and temperance

We assume that the agent's preferences are described by the utility function $U(x)$ defined over the domain $[0,+\infty)$ and continuous in the whole domain. The assumption on the unbounded domain of the utility function is particularly relevant for all results derived in this paper and thus requires a short comment. Obviously, this assumption does not mean that the agent has to deal with a case where consumption is infinite. It means instead that we want to describe the agent's preferences for every possible level of consumption. In fact, although in a specific economic problem the agent's consumption can vary within the closed interval $[\bar{x}, \tilde{x}]$, we usually want to study this problem for every possible value of $\bar{x}$ and $\tilde{x}$, which requires that preferences are defined for the whole set R^{+}.[6] This is also the reason why utility functions which cannot be defined for this whole set are considered less appropriate for economic models studying risk.[7] Finally note that the assumption of a domain $[0,+\infty)$ is crucial in many papers studying desired or appropriate features of the utility function (e.g. Brockett & Golden, 1987; Caballé & Pomansky, 1996; Pratt & Zeckhauser, 1987),[8] it is standard in microeconomic theory (see, for instance, the well-known book by Mas-Colell, Winston, & Green (1995, p. 51)) and it is often introduced in works examining risk problems (recent examples are Çanakoğlu & Özekici, 2010; Choi & Ruszczyński, 2011; Hart, 2011; Roger, 2011).

We assume that the utility function is differentiable four times and we let $\frac{dU(x)}{dx} = U'(x)$, $\frac{d^2U(x)}{dx^2} = U''(x)$, $\frac{d^3U(x)}{dx^3} = U'''(x)$ and

[1] For a very recent discussion on why it is important to study the signs of derivatives of the utility function of order larger than 2 (and in particular of orders 3 and 4) see the recent survey by Eeckhoudt (2012). Many papers (such as those cited in the introduction) study the interpretation of these signs. However they do not analyze the relationship between them.

[2] Note that the opposite is not true: a larger variance does not imply a mean-preserving spread. See also Section 4.

[3] The opposite is not true: larger negative skewness does not imply a mean–variance–preserving contraction. See Section 4.

[4] Again the opposite is not true. See Section 4.

[5] Note that all these recent applications consider the general Nth-degree risk change as defined by Ekern (1980). The cases examined in the present paper are cases where N = 2, 3, 4.

[6] It should also be emphasized that an alternative option could be to assume that $U(x)$ is defined over R instead of R^{+}. However, the domain is often limited to the positive real line excluding negative argument for $U(x)$ in the analysis of consumption, because a negative level of consumption can be difficult to interpret. Note, on the other hand, that the results above hold if the domain of $U(x)$ is R.

[7] This is the case for instance of the quadratic utility which exhibits the problem to be defined only on a bounded interval (see for example Gollier, 2001, p. 27).

[8] The same assumption is also crucial for some conclusions by Kimball (1993) and for the results by Menegatti (2001).