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Invited Review

Zero duality gap in surrogate constraint optimization: A concise review of models

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ABSTRACT

Surrogate constraint relaxation was proposed in the 1960s as an alternative to the Lagrangian relaxation for solving difficult optimization problems. The duality gap in the surrogate relaxation is always as good as the duality gap in the Lagrangian relaxation. Over the years researchers have proposed procedures to reduce the gap in the surrogate constraint. Our aim is to review models that close the surrogate duality gap. Five research streams that provide procedures with zero duality gap are identified and discussed. In each research stream, we will review major results, discuss limitations, and suggest possible future research opportunities. In addition, relationships between models if they exist, are also discussed.

1. Introduction

Consider the optimization problem (P) given as follows where f and each component $g_i(x)$ of the vector function $g(x)$ are real-valued functions defined on X. No specific characteristics of these functions or of X are assumed unless otherwise specified.

$$P : Min_{x\in X} f(x) : \text{st. } g(x) \leqslant 0. \quad (1)$$

Let $X(F)$ be the set of all feasible points in (P) defined by

$$X(F) = \{x \in X : g(x) \leqslant 0\}. \quad (2)$$

No distinction is made between the row and column vectors, and all vector products are dot products in the usual sense and conformable dimensions are taken for granted. Glover (1965) introduced the *surrogate constraint* relaxation as an alternative to the Lagrangian relaxation. A surrogate constraint for problem (P) is a linear combination of the component constraints of $g(x) \leqslant 0$ that associates a multiplier u_i with each $g_i(x) \leqslant 0$ to produce a single inequality $ug(x) \leqslant 0$, where $u = (u_i)$. This inequality is implied by $g(x) \leqslant 0$ whenever $u \geqslant 0$ (Glover, 1975). Given a multiplier vector $u \geqslant 0$, the surrogate problem is then defined by

$$SP(u) : Min_{x\in X} f(x) : \text{st. } ug(x) \leqslant 0. \quad (3)$$

Let the optimal objective function value for $SP(u)$ be $s(u)$ defined by

$$s(u) : inf_{x\in X(u)} f(x) : \text{where } X(u) = \{x \in X : ug(x) \leqslant 0\}. \quad (4)$$

Note that $SP(u)$ is a relaxation of (P), for $u \geqslant 0$, $s(u)$ cannot exceed the optimum objective function value of (P). It approaches this value more closely as $ug(x) \leqslant 0$ becomes a more *'faithful'* representation of the constraint $g(x) \leqslant 0$. Also, we know that $X(F) \subseteq X(u)$ and thus a faithful representation of $g(x) \leqslant 0$ by $ug(x) \leqslant 0$ depends upon how large the set $X(u)$ is compared to the set $X(F)$. Choices of the vector u that improve the proximity of $SP(u)$ to (P), which provide the greatest values of $SP(u)$, yield the *strongest* surrogate constraints. Based upon these facts, the definition of the surrogate dual SD is as follows:

$$SD : Max_{u\geqslant 0}\ s(u). \quad (5)$$

Since $X(F) \subseteq X(u)$, the optimal value of the surrogate dual SD is smaller than or equal to the optimal value of problem (P) and the amount of difference is called the *surrogate duality gap*. The smaller is the gap the more faithfully the single inequality $ug(x) \leqslant 0$ represents the system of inequalities $g(x) \leqslant 0$.

Note that except for non-negativity, there is no restriction on the values of the multiplier vectors u and components u_i of u can be any real number, i.e., integer, rational or irrational (Glover, 1965).

Greenberg and Pierskalla (1970) observed that the optimal value of the SD problem is always as good as the optimal value of the Lagrangian dual (LD) defined by

$$LD : Max_{u\geqslant 0}\ L(u). \quad (6)$$

In (6), $L(u)$ is the function defined by

$$L(u) : inf_{x\in X}\{f(x) + ug(x)\}. \quad (7)$$

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Thus, the surrogate constraint duality gap is always as small or smaller than the Lagrangian duality gap.

Over the years researchers have introduced a variety of different methods for finding the multiplier vectors u that yield the strongest surrogate constraints (see for example, Glover, 1965; Dinkel and Kochenberger, 1978; Dyer, 1980; Karwan and Rardin, 1984; Austin and Ghandforoush, 1985; Gavish et al., 1991; Kim and Kim, 1998; Narciso and Lorena, 1999 and Glover, 2003). An important issue in surrogate constraint optimization is the way that constraints are aggregated to create a single constraint. If it is beneficial to the search process, several constraints may be created instead of a single constraint; however, without the loss of generality we concentrate on the creation of a single constraint. Most constraint aggregation schemes for integer programing, group two constraints into one and then sequentially aggregate each of the remaining constraints with the newly formed constraint. Refer to, Rogers et al. (1991), for a comprehensive survey of aggregation and disaggregation in optimization. Methods for simultaneously combining constraints also have been introduced by several researchers (Rogers et al., 1991 and Glover, 2003). In all of these methods, the concern is to find a set of multipliers to aggregate multiple constraints into a single constraint. If a solution to the single constraint problem also satisfies all constraints of the problem (P) then we have an optimal solution to the original problem which will imply that we have a zero duality gap. An early result to close the duality gap is given by Luenberger (1968) who showed that any quasi-convex programming problem could be solved exactly if the surrogate multipliers are correctly chosen.

Based on the aforementioned discussion, an important research topic is to find a multiplier (or a set of multipliers) u such that the set of solutions to the single constraint $ug(x) \leqslant 0$ is the same as the set of solutions to the system of constraints $g(x) \leqslant 0$. This guarantees $X(F) = X(u)$. Clearly, in order to have the zero surrogate duality gap, it is not necessary to have $X(F) = X(u)$, however, it is a sufficient condition. To the best of our knowledge, there are at least five research streams that attempted to close the duality gap in the surrogate constraint methods. These approaches are categorized as follows: (1) Aggregation of Diophantine equations; (2) Irrational multipliers method; (3) Maximum entropy method; (4) P-norm method; and (5) Slicing algorithm. Rogers et al. (1991) provide a comprehensive discussion regarding surrogate constraint optimization but does not cover several models we discuss here.

In the following sections, major results in each of the five models are reviewed, relationships between models are discussed, limitations are discussed, and possible future research opportunities are suggested.

2. Aggregation of diophantine equations

Finding an aggregation of a given set of equalities to create a single one (or several equalities) with the same set of solutions as the original system of equations possess has been an important research topic over one century. The seminal paper by Mathews (1896), is likely the first publication to provide a solution to this problem. Consider the system of two equations given by (8) where $x_j \geqslant 0$, for $j = 1, \ldots, n$, are unknown integers, and a_{ij} and b_j, for $j = 1, \ldots, n$, and $i = 1, 2$, are given positive integers.

$$\sum_{j=1}^{n} a_{1j}x_j = b_1, \qquad \sum_{j=1}^{n} a_{2j}x_j = b_2, \tag{8}$$

Mathews (1896) showed that the systems of Eq. (8) possess the same set of solutions as the single Eq. (9) provided that u_1 and u_2 are suitably chosen *relatively prime* integers (whose greatest common divisor is one). Furthermore, Mathews extended the result to more than two equations.

$$\sum_{j=1}^{n} (u_1 a_{1j} + u_2 a_{2j})x_j = u_1 b_1 + u_2 b_2. \tag{9}$$

The aggregation of a system of two equations into a single equation is based upon the following property (Anthonisse, 1973).

Proposition 1. *If u_1 and u_2 are two relative prime integers, $u_i \neq 0$, then all integer solutions of the equation*

$$u_1 y_1 + u_2 y_2 = 0, \tag{10}$$

are of the form $y_1 = tu_2$, and $y_2 = -tu_1$ where t is any integer.

Define:

$$y_1 = \sum_{j=1}^{n} a_{1j}x_j - b_1, \qquad y_2 = \sum_{j=1}^{n} a_{2j}x_j - b_2. \tag{11}$$

Now, write the aggregated Eq. (9) in the form of (10). If some 'favorable' assumptions are imposed on Eqs. (8) then by proposition 1 we can choose appropriate relative prime numbers u_1 and u_2 that the set of solutions to the single equation is the same as the set of solutions to the two equations combined. *Any favorable assumption must satisfy that the value of y_2 for all feasible x cannot become a multiple of u_1 for an appropriate choice of u_1*. It follows from Eq. (10). Thus, in this situation it is not possible to find any non-zero number for y_1 or y_2 to satisfy (10). Note that this is also the basis for irrational multipliers method.

Elmaghraby and Wig (1970), used this method for the first time and applied to optimization by adding slack variables to inequalities, making them equalities. Other successful implementations of the method to optimization have been reported in (Glover and Babayev, 1995; Babayev et al., 1997).

During the last several decades researchers have put major emphasis on creating a set of multipliers of u to create a single equation where the set of solutions to the single equation is the same as the set of solutions to the original set of equations. In all such attempts the authors imposed many limiting assumptions, such as: variables and equations must be bounded, variables must be non-negative and integer valued, problem data must be non-negative and integer, equations must be linear with bounded constants (Rogers et al., 1991).

Several successful implementations of the aggregation of equations have been recorded in the literature. Considerable amount of difficulties have also been reported. The difficulties stem from the fact that coefficients of the single constraint can grow very large in practice as the number of equalities increases (Glover and Woolsey, 1972; Kannan, 1983; Fishburn and Kochenberger, 1985 and Khurana and Murty, 2012). It has been shown that the computation required to aggregate constraint equations in a linear integer program with non-negative variables and non-negative coefficients, is polynomial time bounded (Kannan, 1983). Glover and Woolsey (1972), noted that Mathews' method of aggregating sequentially applied to m equations, yields a greater than exponential growth of the coefficients in the resulting constraint.

In the light of these limitations, attempts have been made to generate multipliers that create small coefficients in the equivalent knapsack problem (Rogers et al., 1991). One such attempt is the *log prime* method, proposed by Ram et al. (1988) for linear equalities. In the log prime method, values of multipliers are specific irrational numbers. Alidaee and Wang (2012) have recently proposed a generalization of log prime method. The irrational multipliers method will be discussed in the next section. Now we discuss a successful implementation of Diophantine equations with rational multipliers. This is achieved by transforming the surrogate problem to maximum consistency problem (MCP) introduced by (Glover and Babayev, 1995 and Babayev et al., 1997). The MCP