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Continuous Optimization

An approach for solving a fuzzy multiobjective programming problem

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ABSTRACT

In this paper we present a new approach, based on the Nearest Interval Approximation Operator, for dealing with a multiobjective programming problem with fuzzy-valued objective functions.

By the way we have established a Karush–Kuhn–Tucker (K.K.T) kind of Pareto optimality conditions, for the resulting interval multiobjective program. To this end, we made use of gH-differentiability of involved interval-valued functions.

Two algorithms play a pivotal role in the proposed method. The first one returns a nearest interval approximation to a given fuzzy number. The other one makes use of K.K.T conditions to deliver a Pareto optimal solution of the above mentioned resulting interval program.

1. Introduction

Many decisions that we make in real life cannot be modeled easily in deterministic terms because of imprecision surrounding involved data. In this connection, the noted philosopher Nietzche was quoted as saying "No one is gifted with immaculate perception". This has also been well expressed by the Physics Nobel laureate Feynman who once wrote: "When dealing with a mathematical model, special attention should be paid to imprecision in data". Zadeh 's incompatibility principle [17] stipulating that: "When the complexity of a system increases, our ability to formulate precise and yet meaningful statement on this system decreases up to a threshold beyond which precision and significance become mutually exclusive characteristics", is also instructive in this regards.

This gives substance to the study of mathematical models under uncertainty. As probability theory is a matured segment and a familiar territory of mathematics, it is not a surprise that early works on mathematical programming under uncertainty was devoted to situations where randomness is in the state of affairs [1,28,29,40]. Nevertheless, imprecision cannot be equated with randomness. As a matter of fact, there is a qualitatively different type of imprecision (vagueness) which cannot be tackled with probabilistic apparatus [43]. This has rightly led some researchers to embark upon the investigation of ways of integrating fuzzy relations and/or fuzzy quantities into mathematical programming models [23,34,44].

In this paper, we consider a multiobjective programming problem with fuzzy objective functions.

This is an ill-defined problem. Neither solution concepts (like Pareto Optimality) nor existing approaches (like the weighting method), introduced for deterministic multiobjective programming, can be blindly applied.

For these tools to be applied, they should be properly tailored to take into consideration the fuzziness surrounding the problem.

Moreover, existing approaches for solving the above mentioned problem either caricature the reality or are computationally demanding.

We propose here a new approach that do not give a bad reflect of the reality and in the same time yields a computationally tractable deterministic problem.

At the heart of our approach lies the idea of approximating involved fuzzy quantities by their respective nearest interval approximations. This helps avoiding pitfalls due to severe oversimplification of the reality.

The challenging task of singling out a solution of the resulting interval optimization problem is also addressed.

The remaining of the paper unfolds as follows. In Section 2, we give a brief primer on notions of real interval and fuzzy numbers. In section 3, we discuss the concept of the nearest interval approximation of a fuzzy quantity. In Section 4, we present our approach for dealing with a multiobjective programming problem with fuzzy number coefficients. Section 5 is devoted to a numerical example for the sake of illustration.

In Section 6, we give a critical analysis of existing methods for optimization problems with several fuzzy objective functions. Section 7 is devoted to an assessment of our method in comparison with existing ones.

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We end up, in Section 8 with some concluding remarks along with perspectives for further research in this field.

2. Preliminaries

2.1. Real intervals

We denote by K_C the family of all bounded closed intervals in $\mathbb{R}$; i.e.,

$$K_C = \{[a,b] | a,b \in \mathbb{R} \text{ and } a \leqslant b\}.$$

For $A \in K_C$, we write $A = [a^L, a^U]$ where a^L and a^U are respectively the lower and the upper bounds of A.

The center and the width of an interval $A = [a^L, a^U]$ are respectively given by:

$$a^c = \frac{1}{2}[a^L + a^U]$$
$$a^S = a^U - a^L$$

The generalization of arithmetic in $\mathbb{R}$ to K_C may be found in Moore [36].

The main idea of interval arithmetic is as follows.

Let $* \in \{+, -, \cdot, \div\}$ be a binary operation in $\mathbb{R}$. If A, $B \in K_C$ then

$$A * B = \{a * b \mid a \in A \text{ and } b \in B\} \tag{1}$$

defines a binary operation on K_C.

In the case of division, it is assumed that 0 is not a member of the interval B.

Operations on intervals used in this paper may be explicitly obtained from (1) as follows:

$$A + B = [a^L, a^U] + [b^L, b^U] = [a^L + b^L, a^U + b^U]$$

$$kA = k[a^L, a^U] = \begin{cases} [ka^L, ka^U] & \text{for } k \geqslant 0 \\ [ka^U, ka^L] & \text{for } k < 0 \end{cases}$$

where k is a real number.

In the sequel, we'll also use the generalized Hukuhara difference [7], defined as follows. For A, $B \in K_C$,

$$A \ominus_{gH} B = C \text{ iff} \begin{cases} \text{either } A = B + C \\ \text{or } \quad B = A + (-1)C \end{cases}$$

The gH-difference has many interesting properties, for example $A \ominus_{gH} A = \{0\}$.

Moreover if $A = [a,b]$ and $B = [c,d]$,

$$A \ominus_{gH} B = [\min(a-c, b-d), \max(a-c, b-d)]$$

The following are the two main order relations defined on K_C [25,42].

Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$.

- The order relation $\leqslant_{LU}$ is defined as follows:

$$A \leqslant_{LU} B \text{ if and only if } a^L \leqslant b^L \text{ and } a^U \leqslant b^U \tag{2}$$

- The order relations $\leqslant_{LS}$ and $\geqslant_{LS}$ are defined as follows:

$$A \leqslant_{LS} B \text{if and only if } a^L \leqslant b^L \text{ and } a^S \leqslant b^S \tag{3}$$
$$A \geqslant_{LS} B \text{ if and only if } a^U \geqslant b^U \text{ and } a^S \leqslant b^S \tag{4}$$

The width of an interval can be regarded as an uncertainty (noise), risk, or a type of variance. Therefore, an interval with smaller width and larger upper (smaller lower) bound is considered better for maximization (minimization) purposes.

Proposition 2.1 25. *If $A \leqslant_{LS} B$ then $A \leqslant_{LU} B$*

2.2. Some properties of interval-valued functions

In this paper we consider interval-valued functions of the type: $F: X \subset \mathbb{R}^n \to K_C$.

In the sequel, $F(x)$ is denoted by $[f^L(x), f^U(x)]$.

Let $H: K_C \times K_C \to \mathbb{R}^+$

be given by

$$H(A,B) = \max\{\max_{a \in A} d(a,B), \max_{b \in B} d(b,A)\}$$

where

$$d(a,B) = \min_{b \in B} |a - b|.$$

It is shown in [16] that (K_C, H) is a metric space.

2.2.1. Convexity

Let F be an interval-valued function defined on a convex set $X \subset \mathbb{R}^n$. Then:

(a) F is said to be LU-convex at x^* if

$$F(\lambda x^* + (1-\lambda)x) \leqslant_{LU} \lambda F(x^*) + (1-\lambda)F(x)$$

for all $\lambda \in (0,1)$ and $x \in X$.

(b) F is said to be LS-convex at x^* if

$$F(\lambda x^* + (1-\lambda)x) \leqslant_{LS} \lambda F(x^*) + (1-\lambda)F(x)$$

for all $\lambda \in (0,1)$ and $x \in X$.

The following result, the proof of which may be found elsewhere [4], will be used in the sequel.

Proposition 2.2. *Let X be a convex subset of $\mathbb{R}^n$ and F an interval-valued function defined on X. Then the following properties hold true.*

(a) *F is LU-convex at x^* if and only if f^L and f^U are convex at x^*.*
(b) *F is LS-convex at x^* if and only if f^L and f^U are convex at x^*.*
(c) *If F is LS-convex at x^* then F is LU-convex at x^*.*

2.2.2. Continuity

An interval-valued function F defined on $X \subset \mathbb{R}^n$ is said to be continuous at $x_\circ$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|x - x_\circ\| < \delta$ implies $H(F(x), F(x_\circ)) < \varepsilon$.

If $F(x) = [f^L(x), f^U(x)]$ then the following result the proof of which may be found in [4] holds true.

Proposition 2.3. *Let F be an interval-valued function defined on $X \subset \mathbb{R}^n$ and $x_\circ \in X$. Then F is continuous at $x_\circ$ if and only if f^L and f^U are continuous at x°.*

For more details on continuity of interval-valued functions, the reader may consult [4,13].

2.2.3. gH-differentiability

Consider a real interval T. The gH-derivative of an interval-valued function F: $T \to K_C$ at $t_\circ$ is defined as:

$$F'(t_\circ) = \lim_{h \to 0} \frac{F(t_\circ + h) \ominus_{gH} F(t_\circ)}{h} \tag{5}$$

If $F'(t_\circ) \in K_C$ satisfying (5) exists, we say that F is (gH)-differentiable at $t_\circ$. If F is gH-differentiable at each point $t \in T$, we say that F is gH-differentiable on T.

The generalization of the above definition of gH-derivative to interval-valued function defined on $\mathbb{R}^n$ is as follows.

Let F be an interval-valued function defined on $X \subset \mathbb{R}^n$ and let