European Journal of Operational Research 230 (2013) 321–332

Contents lists available at SciVerse ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

General linear formulations of stochastic dominance criteria

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ARTICLE INFO

Article history:
Received 11 September 2012
Accepted 8 April 2013
Available online 18 April 2013

Keywords:
Stochastic dominance
Utility theory
Non-satiation
Risk aversion
Prudence
Temperance

ABSTRACT

We develop and implement linear formulations of general *N*th order stochastic dominance criteria for discrete probability distributions. Our approach is based on a piece-wise polynomial representation of utility and its derivatives and can be implemented by solving a relatively small system of linear inequalities. This approach allows for comparing a given prospect with a discrete set of alternative prospects as well as for comparison with a polyhedral set of linear combinations of prospects. We also derive a linear dual formulation in terms of lower partial moments and co-lower partial moments. An empirical application to historical stock market data suggests that the passive stock market portfolio is highly inefficient relative to actively managed portfolios for all investment horizons and for nearly all investors. The results also illustrate that the mean–variance rule and second-order stochastic dominance rule may not detect market portfolio inefficiency because of non-trivial violations of non-satiation and prudence.

1. Introduction

Stochastic dominance (SD), first introduced in Quirk and Saposnik (1962), Hadar and Russell (1969) and Hanoch and Levy (1969), is a useful concept for analyzing risky decision making when only partial information about the decision maker's risk preferences is available. The concept is used in numerous empirical studies and practical applications, ranging from agriculture and health care to financial management and public policy making; see, for example, the extensive survey in the text book of Levy (2006). A selection of recent studies in OR/MS journals includes Post (2008), Lozano and Gutiérrez (2008), Blavatskyy (2010), Dupačová and Kopa (2012), Lizyayev and Ruszczyński (2012), Lizyayev (2012) and Brown et al. (2012).

SD imposes general preference restrictions without assuming a functional form for the decision maker's utility function. The SD rules of order one to four are particularly interesting, because they impose (in a cumulative way) the standard assumptions of non-satiation, risk-aversion, prudence and temperance, which are necessary conditions for standard risk aversion (Kimball, 1993). This approach is theoretically appealing but not always easy to implement. In some special cases, a closed-form analytical solution exists, as is true, for example, for the textbook case of a pair-wise comparison of two given prospects based on the second-order stochastic dominance (SSD) rule.

However, more generally, a closed-form solution does not exist and numerical optimization is required. For example, Meyer's (1977a,b) stochastic dominance with respect to a function (SDWRF) requires solving an (small and standard) optimal control problem. The rules of convex stochastic dominance (Fishburn, 1974) for comparing more than two prospects simultaneously also require optimization. For example, Bawa et al. (1985) develop Linear Programming tests for convex first-order stochastic dominance (FSD), convex SSD and an approximation for convex third-order stochastic dominance (TSD). Shalit and Yitzhaki (1994), Post (2003), Kuosmanen (2004) and Kopa and Chovanec (2008) develop Linear Programming tests that compare a given prospect using SSD with a polyhedral set of linear combinations of a discrete set of prospects.

Unfortunately, a general algorithm is not available. How can we test, for example, whether a given medical treatment is dominated by convex fourth-order stochastic dominance (FOSD) relative to a set of alternative treatments? How can we test whether a given investment portfolio is FOSD efficient relative to a polyhedral set of portfolios formed from a set of base assets? Without an algorithm for these specific cases, we may be forced to use known tests for less discriminating decision criteria. For example, we could use a set of pair-wise FOSD tests to compare the evaluated medical treatment with every alternative treatment. Similarly, we could use pair-wise tests to compare the evaluated investment portfolio with a large number of alternative portfolios, for example, using a grid search or random search over the possibilities set. However, pair-wise comparisons generally are less powerful than convex SD, because a prospect can be non-optimal for all admissible utility functions without being dominated by any alternative prospect. A

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http://dx.doi.org/10.1016/j.ejor.2013.04.015

further possible loss of power stems from using a discrete approximation to a continuous choice set.

Section 2 of this study develops linear formulations of general stochastic dominance rules. Our approach is based on a piece-wise polynomial representation of utility and its derivatives. This representation applies generally for higher-order SD rules (*N*th order SD), comparing a given prospect with a discrete set of alternative prospects (convex NSD analysis), and comparing a given prospect with a polyhedral set of linear combinations of prospects (NSD efficiency analysis). Our analysis therefore represents a generalization of the lower-order tests of Bawa et al. (1985) and Post (2003). We can also deal with additional preference restrictions such as the bounds on the level of risk aversion of Meyer (1977a,b) and the bounds on utility curvature by Leshno and Levy (2002). The use of piece-wise polynomial functions also generalizes results by Hadar and Seo (1988) and Russell and Seo (1989) on simple representative utility functions for pairwise comparison based on lower-order SD rules.

To arrive at a finite optimization problem, we focus on discrete probability distributions. In empirical studies, we usually face discrete sample distributions, and experimental studies generally use prospects with a discrete population distribution. In addition, many continuous distributions can be approximated accurately with a discrete distribution. Our approach can be implemented by solving a relatively small system of linear inequalities. The linear structure seems particularly convenient for the application of statistical re-sampling methods in the spirit of Nelson and Pope (1991) and Barrett and Donald (2003).

Our focus is on utility and its derivatives and on restrictions that follow from utility theory. Still, Section 3 also derives linear dual formulations that are formulated in terms of lower partial moments (Bawa, 1975) and co-lower partial moments (Bawa and Lindenberg, 1977) of the probability distribution. We focus on the dominance classification of a given prospect and we do not attempt to identify an alternative prospect that dominates the evaluated prospect. In the case of a discrete choice set, a non-admissible prospect need not be dominated by any alternative prospect. In addition, a prospect that dominates the choice of a given decision maker need not be optimal for that decision maker, and, moreover, the optimum need not dominate the current choice. Finally, the dominance relation between a pair of prospects generally is less robust than the classification of a given prospect. For these reasons, the search for a dominant prospect seems irrelevant for our purposes. Still, the dual formulations are useful for computational efficiency and robustness analysis.

Section 4 applies a range of SD tests to historical stock return data to compare the broad stock market portfolio with alternative portfolios formed from a set of risky benchmark stock portfolios and riskless Treasury bills. We analyze horizons ranging from 1 month to 10 years and consider the decision criteria of SSD, TSD, FOSD, SDWRF, ASSD and mean–variance (M–V) analysis. The analysis is relevant because a large class of capital market equilibrium models predict that the market portfolio is efficient. Another reason for expecting market portfolio efficiency is the popularity of passive mutual funds and exchange traded funds that passively track broad stock market indices.

Our empirical analysis shows that the market portfolio is highly and significantly inefficient by the TSD, FOSD, SDWRF and ASSD criteria for every horizon. Few rational risk averters would hold the broad market portfolio in the face of the historical return premiums to active strategies. The appeal of active strategies only increases with the horizon. Our results also illustrate that pair-wise dominance comparisons and the SSD and M–V rules have limited discriminating power and can generate misleading results in relevant applications. The SSD criterion may fail to detect market portfolio inefficiency for short horizons, because it penalizes small-cap stocks for having a relatively high positive systematic skewness, violating prudence. M–V analysis underestimates the level of market portfolio inefficiency for long horizons, because it assigns negative weights to large positive market returns, placing a penalty on outperformance during bull markets. In our application, these phenomena lead to a non-trivial underestimation of the alphas for small-cap stocks.

2. Linear formulation in terms of piece-wise polynomial utility

We consider *M* prospects with risky outcomes $x_1,\ldots,x_M$. A prospect is defined here in a general way as an available choice alternative and it could be a given combination of multiple base alternatives, for example, a combination of production methods, financial assets or marketing instruments. Depending on the application, the outcomes may be total wealth, consumption, income, or any variable that can reasonably be assumed to enter as an argument to a utility function that obeys the maintained assumptions. The outcomes are treated as random variables with a discrete, state-dependent, joint probability distribution characterized by *R* mutually exclusive and exhaustive scenarios with probabilities $p_r > 0$, $r = 1,\ldots,R$. We use $x_{i,r}$ for the outcome of prospect *i* in scenario *r*. We collect all possible outcomes across prospects and states in $Y = \{y\colon\ y = x_{i,r}\ \ i = 1,\ldots,M;\ r = 1,\ldots,R\}$, rank these values in ascending order $y_1 \leqslant \cdots \leqslant y_S$ and use $q_{i,s} = \Pr[x_i = y_s] = \sum_{r:x_{i,r}=y_s} p_r$.

Decision makers' preferences are described by *N*-times continuously differentiable, von Neumann–Morgenstern utility functions $u(x)\colon D \to \mathbb{R}$. We use $u^n(x)$ for the *n*th order derivative, $n = 1,\ldots,N$, and $u^0(x) = u(x)$. To implement stochastic dominance of order $N \geqslant 1$, we will consider the following set of admissible utility functions:

$$U_N = \{u \in C^N :\ (-1)^{n-1}u^n(x) \geqslant 0 \quad \forall x \in D,\ n = 1,\ldots,N\}. \tag{1}$$

Thus, first-order dominance assumes non-satiation ($u^1(x) \geqslant 0$, $\forall x \in D$); second-order dominance assumes also risk aversion ($u^2(x) \leqslant 0,\ \ \forall x \in D$); the third-order criterion adds prudence ($u^3(x) \geqslant 0$, $\forall x \in D$) and fourth-order SD also assumes temperance ($u^4(x) \leqslant 0$, $\forall x \in D$). In some applications, zero values for the derivatives may not be allowed, for example, in the cases of strict non-satiation ($u^1(x) > 0,\ \forall\ x \in D$) and strict risk aversion ($u^2(x) < 0$, $\forall x \in D$). The needed adjustments to our Linear Programming tests are obvious substitutions of weak and strict inequalities. In our experience, these adjustments have a negligible effect in empirical applications. For the sake of brevity, we therefore ignore this issue here.

For practical reasons, it is often useful to assume some sort of standardization, such as $u^1(y_1) = 1$, in order to avoid numerical problems when evaluating utility functions that approximate $u^1(x) = 0\ \ \forall x \in D$, or the indifferent decision maker. Since utility analysis is invariant to positive linear transformations, such standardizations are harmless.

We distinguish between three types of SD relations: pair-wise dominance relations, discrete convex dominance relations and continuous convex dominance relations, or efficiency relations. These relations differ regarding to the assumed choice possibilities: a single prospect, a discrete set of prospects, or all convex combinations of a discrete set of prospects. Consider first the case of pair-wise comparison between two given prospects:

Definition 1 (*Pair-wise Comparison*). An evaluated prospect $i \in \{1,\ldots,M\}$ is not dominated in terms of *N*th order stochastic dominance, $N \geqslant 1$, by an alternative prospect $j \in \{1,\ldots,M\}$ if there exists an admissible utility function $u \in U_N$ for which it is preferred to the alternative: