European Journal of Operational Research 228 (2013) 397–404

Contents lists available at SciVerse ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

Global maximization of UTA functions in multi-objective optimization

Duy Van Nguyen

Department of Mathematics, University of Trier, D-54286 Trier, Germany

ARTICLE INFO

Article history:
Received 5 May 2011
Accepted 12 June 2012
Available online 7 July 2012

Dedicated to the memory of my teacher, Prof. Dr. Reiner Horst.

Keywords:
Multiple objective optimization
Utility function program
UTA type methods
UTA functions
Global optimization
Branch-and-bound algorithm

ABSTRACT

The UTAs (UTilité Additives) type methods for constructing nondecreasing additive utility functions were first proposed by Jacquet-Lagrèze and Siskos in 1982 for handling decision problems of multicriteria ranking. In this article, by UTA functions, we mean functions which are constructed by the UTA type methods. Our purpose is to propose an algorithm for globally maximizing UTA functions of a class of linear/convex multiple objective programming problems. The algorithm is established based on a branch and bound scheme, in which the branching procedure is performed by a so-called *I*-rectangular bisection in the objective (outcome) space, and the bounding procedure by some convex or linear programs. Preliminary computational experiments show that this algorithm can work well for the case where the number of objective functions in the multiple objective optimization problem under consideration is much smaller than the number of variables.

1. Introduction

The subject for this article is the multiple objective programming problem of the form

$$\begin{aligned} &\max\ f_i(x),\ i=1,\cdots,k \\ &\text{s.t.}\quad x\in X\subset\mathbb{R}^n. \end{aligned} \tag{1}$$

We assume throughout the article that $k\geqslant 2$, X is a compact convex set and the functions $f_i : X\to\mathbb{R},\ i=1,\cdots,k$, are linear or concave. Furthermore, we define the mapping

$$f:\mathbb{R}^n\to\mathbb{R}^k,\quad f(x)=(f_1(x),\cdots,f_k(x)).$$

Usually, X is called the decision set, and the image of the set X under f,

$$f(X):=\{z\in\mathbb{R}^k:\ z=f(x),\ x\in X\}, \tag{2}$$

is called the outcome set of Problem (1).

The most important approach for handling Problem (1) is the concept of efficient (or Pareto optimal) solutions. For applications, theory and computational methods of efficient solutions, we refer to Refs. [1–8,18,21–23] and references therein.

Another interesting and important approach to Problem (1) is the concept of utility (or value) functions and utility function programs. The main idea of this school of thought is that each outcome has a 'utility' to the decision maker, and the decision maker tends to or should choose the one having the maximal utility. This approach consists of two consecutive stages:

(a) Construction of a utility function on the outcome set, and
(b) Development of algorithms for the global maximization of the utility function.

For the first stage, the UTAs (UTilité Additives) type methods, originally proposed by Jacquet-Lagrèze and Siskos [13] for handling multicriteria decision making problems, play an important part. These methods construct utility functions of the form

$$U(z)=\sum_{i=1}^{k}U_i(z_i),$$

where for each i, U_i is a continuous, nondecreasing and piecewise affine univariate function. Throughout this article, functions constructed by the UTA type methods are called UTA functions.

For improvements of the UTA type methods and other methods for constructing utility functions we refer to e.g., [4,6–8,14–17,19,20,22,23].

For the second stage, to our knowledge, there are only a few methods for handling utility function programs. In [12], a modification of the simplex algorithm was proposed to determine some kind of local optimal solutions. A global optimization method for solving general utility function programs was presented in [10].

The purpose of the present article is to propose a new algorithm for globally solving the utility function program

E-mail address: nguyen@uni-trier.de

0377-2217/$ - see front matter © 2012 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.ejor.2012.06.022

$$\max\{U(z) : z \in f(X)\} = \max\{U(z) : z = f(x), x \in X\}, \tag{3}$$

where $U(z)$ is a UTA function. Our algorithm belongs to the class of branch and bound methods, which is a successful tool in global optimization, see e.g., [9–11]. In this algorithm, both basic operations, branching and bounding procedures, are established based on the special structure of the UTA function $U(z)$.

To apply our algorithm, we assume that any UTA function, constructed by the basic model of Jacquet-Lagrèze, and Siskos, or by the new approaches, e.g., the UTA–GMS method in [7], the GRIP method in [6], the ACUTA method in [4], is consistent by applying a man–machine dialogue included in the UTA type philosophy.

In the next section, we outline briefly the basic UTA method and give a detailed representation of the resulting UTA function. Sections 3–5 deal with basic operations that are used to establish the finite branch and bound algorithm in Section 6. An illustrative example and preliminary computational experiments are reported in Section 7. Finally, some conclusions are given in Section 8.

2. The Basic UTA Method

The basic UTA method of Jacquet-Lagrèze and Siskos was developed for constructing additive utility functions of the form

$$U(z) = \sum_{i=1}^{k} U_i(z_i), \tag{4}$$

where for every i, the univariate function $U_i(z_i)$ is continuous, nondecreasing and piecewise affine in the interval $[t_i, T_i]$ with

$$t_i \leqslant \min\{f_i(x) : x \in X\} < \max\{f_i(x) : x \in X\} \leqslant T_i. \tag{5}$$

For arbitrary points $z^1, z^2 \in \mathbb{R}^k$, by $z^1 \leqslant z^2$ we mean that $z^1_i \leqslant z^2_i \ \forall i = 1, \cdots, k$. Let R^0 be a rectangle in $\mathbb{R}^k$ defined by

$$R^0 := \{z \in \mathbb{R}^k : t \leqslant z \leqslant T\} = \{z \in \mathbb{R}^k : t_i \leqslant z_i \leqslant T_i, \ i = 1, \cdots, k\}. \tag{6}$$

Then obviously the function U defined in (4) has the following nondecreasing property in the rectangle R^0:

$$z^1, \quad z^2 \in R^0, \quad z^1 \geqslant z^2 \Rightarrow U(z^1) \geqslant U(z^2). \tag{7}$$

We present briefly the original idea of the basic UTA method.

UTA Method:

Step 1: Choose a finite set, B, of reference outcomes, (e.g., B consists of outcomes of efficient solutions).

Step 2: Determine on B a preference ordering in the following sense: for each pair $a, b \in B$, one is convinced that exactly one of following cases can occur:
(i) a is preferred to b, notation: $a \succ b$;
(ii) a is indifferent to b, notation: $a \sim b$.

Step 3: Divide interval $[t_i, T_i]$ into $\pi_i - 1$ equal intervals $[z_{i,j}, z_{i,j+1}]$, $j = 1, \cdots, \pi_i - 1$, with $z_{i1} = t_i$ and $z_{i,\pi_i} = T_i$.

Step 4: For each i, assume that $U_i(z_i)$ is continuous and affine in each subinterval $[z_{i,j}, z_{i,j+1}]$, $j = 1, \cdots, \pi_i - 1$. From this, for each $b \in B$ and $i = 1, \cdots, k$, there is an interval $[z_{i,j}, z_{i,j+1}]$ such that $b_i \in [z_{i,j}, z_{i,j+1}]$, and it holds that

$$U_i(b_i) = U_i(z_{i,j}) + \frac{U_i(z_{i,j+1}) - U_i(z_{i,j})}{z_{i,j+1} - z_{i,j}}(b_i - z_{i,j}). \tag{8}$$

Step 5: Assume that for each $b \in B$, the function value $U(b)$ is estimated by

$$U(b) = \sum_{i=1}^{k} U_i(b_i) + \varepsilon_b, \tag{9}$$

where $U_i(b_i)$ is computed by (8), and ε_b denotes an error of the estimation. Furthermore, assume that for each pair $a, b \in B$, it holds that

$$\left.\begin{aligned} a \sim b &\Longleftrightarrow U(a) - U(b) = 0 \\ a \succ b &\Longleftrightarrow U(a) - U(b) \geqslant \delta, \end{aligned}\right\} \tag{10}$$

where δ is a given positive number, (a potential error).

Step 5: Denote

$$u_{i,j} = U_i(z_{i,j}), \ i = 1, \cdots, k; \ j = 1, \cdots, \pi_i. \tag{11}$$

Then the values $u_{i,j}$ are determined by solving the following linear program:

$$\min \sum\nolimits_{b \in B} \varepsilon_b \text{ subject to} \tag{12}$$

$$\begin{gathered} \sum_{i=1}^{k} [U_i(a_i) - U_i(b_i)] + \varepsilon_a - \varepsilon_b \geqslant \delta \\ \text{for all pair } a, b \in B \text{ such that } a \succ b \end{gathered} \tag{13}$$

$$\begin{gathered} \sum_{i=1}^{k} [U_i(a_i) - U_i(b_i)] + \varepsilon_a - \varepsilon_b = 0 \\ \text{for all pair } a, b \in B \text{ such that } a \sim b \end{gathered} \tag{14}$$

$$u_{i,j+1} - u_{i,j} \geqslant 0 \ \forall i = 1, \cdots, k; \ j = 1, \cdots, \pi_i. \tag{15}$$

$$\begin{gathered} u_{i1} = 0 \ \forall i = 1, \cdots, k \\ \sum_{i=1}^{k} u_{i,\pi_i} = 1 \end{gathered} \tag{16}$$

$$u_{i,j} \geqslant 0 \ \forall (i,j), \ \varepsilon_a \geqslant 0 \ \forall a \in B. \tag{17}$$

Remark 1.

(a) The above linear program has $\sum_{i=1}^{k} \pi_i + |B|$ variables. They are $u_{i,j}$, $i = 1, \cdots, k$; $j = 1, \cdots, \pi_i$ and ε_b, $b \in B$.
(b) Constraints in (13) and (14) are equivalent to conditions in (10).
(c) Constraints in (15) imply that the function $U_i(z_i)$ is nondecreasing for each i.
(d) Constraints in (16) imply that the function $U(z)$ is normed in the rectangle $\{z \in Z : t_i \leqslant z_i \leqslant T_i, i = 1, \cdots, k\}$, such that $U(t) = 0$ and $U(T) = 1$.

Let $u^* = \left(u^*_{ij}\right)_{i=1,\cdots k; j=1,\cdots,\pi_i}$ be an optimal solution of the linear program constructed in Step 5. Then we obtain for each $i \in \{1, \cdots, k\}$ the function $U_i(z_i)$, which is a nondecreasing, continuous and piecewise affine function given by

$$U_i(z_i) = \ell_{ij}(z_i) \text{ for } z_i \in [z_{i,j}, z_{i,j+1}], \quad j = 1, \cdots, \pi_i - 1, \tag{18}$$

where for each $j = 1, \cdots, \pi_i - 1$ the affine function ℓ_{ij} is defined by (8), i.e.,

$$\begin{gathered} \ell_{ij}(z_i) = \alpha_{ij} z_i + \beta_{ij} \text{ with} \\ \alpha_{ij} = \frac{u^*_{i,j+1} - u^*_{i,j}}{z_{i,j+1} - z_{i,j}} \quad \text{and} \quad \beta_{ij} = u^*_{i,j} - \frac{u^*_{i,j+1} - u^*_{i,j}}{z_{i,j+1} - z_{i,j}} z_{i,j}. \end{gathered} \tag{19}$$

3. Reformulation of utility function program

Let U be a UTA function defined in the rectangle R^0, (recall that the rectangle R^0 is defined by (6)). For the establishment of a branch and bound algorithm for solving the utility function program (3), we define a set $Z \subset \mathbb{R}^k$ by