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Discrete Optimization

Variable neighbourhood structures for cycle location problems

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ABSTRACT

Variable neighbourhood search is a metaheuristic used mainly to tackle combinatorial optimization problems. Its performance depends on having a good variable neighbourhood structure: that is, a sequence of neighbourhoods that are ideally pairwise disjoint and contain feasible solutions further and further from a given feasible solution. This article defines a variable neighbourhood structure with these properties that is new for cycle location problems. It find bounds for the neighbourhood sizes and shows how to iterate over then when the cycle is a circuit. It tests the structure and iteration method using variable neighbourhood search on a range of median cycle problems and finds a neighbourhood size beyond which there is, on average, no benefit in applying local search. This neighbourhood size is found not to depend on problem size or bound on circuit length.

1. Introduction

The purpose of this article is to construct a well-defined neighbourhood structure for cycle location problems, to show how to iterate over each neighbourhood $N_k(C)$ of a cycle C, and to investigate how this neighbourhood structure might be effectively used in *variable neighbourhood search* (VNS) and related metaheuristics.

Many problems in location–allocation and scheduling can be expressed in the following form. Given a structure E, a set $\mathcal{F}$ of feasible substructures of E and a function $f : \mathcal{F} \to \mathbb{R}$, find $C \in \mathcal{F}$ to minimise $f(C)$. For example, in the travelling salesman problem (TSP) E is the set of edges of a graph, $\mathcal{F}$ is the set of Hamilton tours and $f(C)$ is the sum of the lengths of the edges of C. In the p-median problem (see Hansen and Mladenović, 1997) E is the set of nodes of a graph, $\mathcal{F}$ is the set of all subsets of E of cardinality p and $f(C)$ is the sum over all $v \in E$ of the minimum distance from v to a node of C. In problems requiring a subgraph of a given graph, the subgraph is usually considered an *extended facility* and the problems usually called *location* problems (Mesa and Boffey, 1996).

Typically such problems are NP-hard and a common tactic to find a good solution is to use some heuristic to look for improving feasible solution sets. In general it is too expensive to search all of $\mathcal{F}$ and so, given a feasible solution $C \in \mathcal{F}$, heuristics typically define $N(C) \subseteq \mathcal{F}$, a *neighbourhood* of C, in which to look for new and usually improving feasible solutions. $N(C)$ is usually defined so that $C' \in N(C)$ is in some sense close to C.

Usually there is more than one way to define $N(C)$. A well-known example is that of the 2-opt and 3-opt neighbourhoods used for the TSP (Cook et al., 1998) and vehicle routing problems (Nagy and Salhi, 2005). The 2-opt neighbourhood of C comprises (for the TSP) tours obtained by removing two edges of C and inserting two new ones. The 3-opt neighbourhood is similar but three edges are removed and three new ones inserted. Section 4 discusses these as exchange operations.

We can search smaller neighbourhoods faster but larger neighbourhoods are more likely to contain an improving solution. Variable neighbourhood search (Mladenović and Hansen, 1997; Hansen and Mladenović, 2001; Hansen et al., 2010) seeks a compromise between speed and likelihood of improvement by defining a *neighbourhood structure* $N_1, \ldots, N_{\max}$. Given $C \in \mathcal{F}$ VNS seeks an improving solution first in $N_1(C)$ and seeks an improving solution in $N_k(C)$ $(k > 1)$ only if no improving solution is found in $N_{k-1}(C)$. Usually $N_1, \ldots, N_{\max}$ are pairwise disjoint and chosen so that, in some sense, a set in $N_k(C)$ is further from C than one in $N_{k-1}(C)$. VNS has been used for many problems including: p-median problems (Hansen and Mladenović, 1997; Fathali and Taghizadeh Kakhki, 2006; Fleszar and Hindi, 2008; Ilić et al., 2010) cycle location problems such as the median cycle problem (Moreno Pérez et al., 2003), TSP and related problems (Felipe et al., 2009; da Silva and Urrutia, 2010; Parragh et al., 2010; Schilde et al., 2011) and vehicle routing problems (Bräysy, 2003; Kytöjokia et al., 2007; Paraskevopoulos et al., 2008; Hemmelmayr et al., 2009); and various assignment, scheduling and allocation problems (Yagiura et al., 1998; Hansen et al., 2007; Lusa and Potts, 2008; Wang and Tang, 2009; Perez-Gonzalez and Framinan, 2010).

Section 2 provides an example to show why it is useful to seek neighbourhood structures with particular properties rather than define them *ad hoc* by a convenient set of operations. Section 3 introduces some notation, and describes cycles and cycle location problems. It also defines a general neighbourhood structure

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$N_1,\ldots,N_{\max}$ where max may be chosen so that the disjoint union of $N_1(C),\ldots,N_{\max}(C)$ is $\mathcal{F}\setminus\{C\}$, and shows that this structure is well defined for cycle location problems. Section 4 identifies, when $\mathcal{F}$ comprises circuits, the operations that compose each neighbourhood in this neighbourhood structure. Section 5 shows how to iterate over each $N_k(C)$, and Section 6 investigates the performance of VNS on a range of cycle median problems for various values of max.

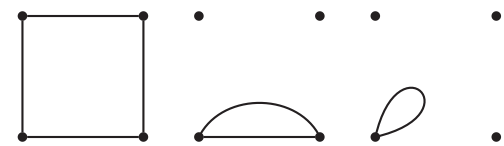

Fig. 1. Circuits of a graph.

2. A motivating example

To see why it might be useful to investigate variable neighbourhood structures for cycle location problems, it is useful to contrast what is known about the best-defined neighbourhood structures with those that have been used for cycle location problems. Hansen and Mladenović (1997) describe the following properties for the p-median problem, though any problem for which $\mathcal{F}=\{C\subseteq E:|C|=p\}$ for some fixed p has the same properties. For C, $C'\in\mathcal{F}$,

$$\rho(C,C')=|C\setminus C'|=|C'\setminus C|, \tag{1}$$

is a convenient definition of distance between two feasible solutions and

$$N_k(C)=\{C'\in\mathcal{F}:\rho(C',C)=k\}, \tag{2}$$

defines a neighbourhood structure $N_1,\ldots,N_{\max}$. $N_1(C),\ldots,N_{\max}(C)$ are pairwise disjoint and if $\max=\min(p,|E|-p)$ then $\cup_{k=1}^{\max}N_k(C)=\mathcal{F}\setminus\{C\}$. Hansen and Mladenović (1997) show that $|N_k(C)|=\binom{p}{k}\binom{n-p}{k}$, and it is straightforward to derive from their proof of this a method of iterating over any given $N_k(C)$ in much the same way Section 5 derives methods for iterating over $N_k(C)$ when C is a circuit.

Hansen and Mladenović (1997) also show that for the p-median problem $N_k(C)$ can be defined by an interchange operation. Hansen et al. (2010) recommend defining neighbourhood structures using operations that produce small perturbations of an existing solution, and the literature sometimes conflates the terms *operations* and *neighbourhood structure*. However, operations do not, in general, define a neighbourhood structure with mutually exclusive neighbourhoods and a well-defined distance function. In particular, the neighbourhood structures that have been used for cycle location problems allow only a small range of neighbourhoods, with the larger ones defined *ad hoc*.

3. Cycles and neighbourhoods

This section defines cycles and circuits so that it may describe some cycle location problems. It then defines a general neighbourhood structure using a symmetric difference operator.

3.1. Definitions

This article discusses properties of a graph G on n nodes with a weight function $w:E(G)\to[0,\infty)$ giving the length of each edge. It uses the following definitions. An *even subgraph* is a graph with no isolated nodes, each of whose nodes has even degree. A *cycle* is a connected even subgraph. A *circuit* is a connected graph each of whose nodes has degree two. A path (from u to v) is a graph that may be made into a circuit by adding a single edge uv joining nodes u and v. Given a even subgraph, cycle, circuit or path C, $|C|$ is shorthand for $|E(C)|$, and the *length* of C is $w(C)=\sum_{e\in C}w(e)$.

Note that a cycle or path need not contain any edges, and a circuit may contain a loop or two parallel edges.

A cycle location problem is one that requires us to locate a cycle C satisfying some constraints defined by G and usually minimising some objective function. Typically we require G to be connected and C to be either (i) a subgraph of G that is a cycle, or (ii) a circuit containing one edge of G twice. Condition (ii) is not usually explicitly given in the literature. But our definition of a cyle permits only solutions with one or with three or more nodes. In practice, solutions with two nodes are both acceptable and useful as neighbours in a metaheuristic. So, define a *cycle of G* to be a graph satisfying condition (i) or (ii).

Similarly it is useful to allow circuits with fewer than three nodes. So, define a *circuit of G* to be either a cycle of G that is a circuit, or a node u of G together with a (loop) edge uu with $w(uu)=0$. Fig. 1 illustrates the possibilities for circuits.

The following definitions are needed to describe examples of cycle location problems. The *distance* between nodes u and v of G is $w(u,v)=\min_P w(P)$, where the minimum is taken over all paths P from u to v. Then, provided $V(C)\subseteq V(G)$, we can define for a subgraph C, the *distance sum*

$$\mathrm{ds}(C)=\sum_{u\in V(G)}\min_{v\in V(C)}w(u,v),$$

and the *eccentricity*

$$\mathrm{ec}(C)=\max_{u\in V(G)}\min_{v\in V(C)}w(u,v).$$

These definitions give a range of problems. The TSP is to find a circuit C of G minimising $w(C)$ subject to $\mathrm{ds}(C)=0$ and $|C|=n$. (The second condition may be dropped if $w>0$.) The covering salesman problem (Current and Schilling, 1989) is to find a circuit C of G minimising $w(C)$ subject to a bound on ec(C). The median cycle and cycle centre problems (Foulds et al., 2004; Labbé et al., 2005) require a circuit of G that minimises distance sum or eccentricity subject to a bound on length. Problems requiring a path between nodes u and v can be cast as circuit location problems by imposing an edge of weight 0 between u and v. Laporte and Rodríguez Martín (2007) discuss a range of further problems, including vehicle routing problems, in which C may be a cycle and may contain specified nodes or be required to satisfy further constraints.

The following observation (see Cook et al., 1998, Chapter 7) is well known. If G' is a connected graph with weight function w', we can define a complete graph G on $V(G')$ with weight $w(uv)=w'(u,v)$ whenever u, $v\in V(G')$. Very often a problem that requires a cycle of G' is easily transformed to an equivalent problem that requires a circuit of G. So this article considers only problems requiring a circuit of a complete graph. It is likely that the results generalise to problems such as vehicle routing, where more than one circuit is required. Notice that the construction of G from G' implies the triangle inequality,

$$w(tv)\leqslant w(tu)+w(uv). \tag{3}$$

Section 6 assumes this inequality holds.

3.2. A general neighbourhood structure

If the feasible solutions to some problem are well defined by sets or even subgraphs, then the following approach produces a well-defined general neighbourhood structure.

Let Δ denote the symmetric difference operator: that is, for sets C and C', $C\,\Delta\,C'=(C\cup C')\setminus(C\cap C')$. Since any even subgraph (or