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Discrete Optimization

A polynomial time algorithm to solve the single-item capacitated lot sizing problem with minimum order quantities and concave costs

Bertrand Hellion *, Fabien Mangione, Bernard Penz

Université de Grenoble/Grenoble-INP/UJF-Grenoble 1/CNRS, G-SCOP UMR5272 Grenoble, F-38031, France

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ABSTRACT

This paper deals with the single-item capacitated lot sizing problem with concave production and storage costs, and minimum order quantity (CLSP-MOQ). In this problem, a demand must be satisfied at each period t over a planning horizon of T periods. This demand can be satisfied from the stock or by a production at the same period. When a production is made at period t, the produced quantity must be greater to than a minimum order quantity (L) and lesser than the production capacity (U). To solve this problem optimally, a polynomial time algorithm in $O(T^5)$ is proposed and it is computationally tested on various instances.

1. Introduction

This paper deals with a generalization of the single-item capacitated lot sizing problem (CLSP) with fixed capacity. The CLSP consists in satisfying a demand at each time period t over a planning horizon T. The demand is satisfied from the stock or by a production. Costs incur for each production and when an item is stored between two consecutive periods. A fixed maximum production capacity (U) must be respected. The problem we consider in this paper contains a minimum order quantity constraint. This constraint imposes that if a production is done at a given period, the quantity must be greater to or equal than a minimum level L. The U and L values are constant for the T periods. This problem is noted CLSP-MOQ in the following:

The single-item capacitated lot sizing problem is known to be *NP*-Hard [2]. However, some cases are polynomially solvable. This is the case when the capacity is fixed over the T periods. Florian and Klein [6] considered a case where production and holding cost functions are concave. They proposed an exact method with a time complexity in $O(T^4)$. Later van Hoesel and Wagelmans [12] improved the complexity of the algorithm in $O(T^3)$ when the holding costs are linear. A complete survey on the single-item lot sizing problem can be found in [3].

The CLSP-MOQ is relevant in some industrial contexts. Lee [8] studied an industrial problem where a manufacturer imposes a minimum order quantity to its supplier. The author took an example where the buyer has to choose a supplier among a manufacturer using MOQ constraints and a local dealer. The local dealer supplies the products in just-in-time with a higher cost per unit. The author designed an $O(T^4)$ algorithm which has been tested on industrial data. Porras and Dekker [11] studied an industrial case where the producer imposes minimum order quantities (MOQ) to produce the items. The company uses containers to ship the products, and set-up costs were not specified explicitly. Consequently, in order to save fixed costs, the producer imposes the MOQ constraint, which plays the role of set-up cost. Zhou et al. [13] analyzed a class of simple heuristic policies to control stochastic inventory systems with MOQ constraints. They also developed insights into the impact of MOQ constraints on repeatedly ordered items to fit in an industrial context.

The first studies on the MOQ constraints were from Constantino [5] and Miller [9]. They analyzed these constraints from a polyhedral point of view. Constantino derived strong inequalities which described the convex hull of the solutions, considering the production level as a continuous variable. Miller [9] replaced the production level by the amount of product that is produced in excess of the lower bound. Thus he studied the facets of the solutions's convex hull. He focused on multi-item problems and he proved that the single period relaxation is NP-hard. Chan and Muckstadt [4] studied a production-inventory system in which the production quantity is constrained by a minimum and a maximum level in each period. However, the production level cannot be zero. They characterized the optimal policy for finite and infinite time horizons. The first exact polynomial time algorithm was recently developed by Okhrin and Richter [10]. They solved a special case of the problem in which the unit production cost is constant over the whole horizon and then can be discarded. Furthermore, they assumed that the holding costs are also constant over the T periods. Considering this restriction, they derived a polynomial time

* Corresponding author.
E-mail address: bertrand.hellion@grenoble-inp.fr (B. Hellion).

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algorithm in $O(T^3)$. Li et al. [7] studied the single item lot sizing problem with lower bound and described a polynomial algorithm in $O(T^7)$ to solve the special case with concave production and storage cost function.

In this study, we extend Okhrin and Richter's result [10] to the problem with concave production and holding costs. We proposed an optimal algorithm called *HMP* with a time complexity in $O(T^5)$. The paper is organized as follows: Section 2 describes the problem and introduces the notations. In Section 3, we give some definitions and present the properties that allow us to solve the problem in polynomial time. The algorithm and its time complexity study are given. In Section 4, the efficiency of the method is tested on various instances. Finally, concluding remarks and perspectives are given in Section 5.

2. Problem description and notations

The single-item lot sizing problem consists in satisfying the demands over T consecutive periods. At each period t, the demand d_t must be satisfied by production at period t (X_t) and/or from the inventory available at the end of the period $t-1$ (I_{t-1}).

The production at each period is constrained by a constant capacity U. If a production is done at period t, it must be greater than or equal to a non-zero minimum order quantity L. We also consider production and inventory costs. The production cost is a concave function of the quantity produced ($p_t(X_t)$) and the inventory cost is a concave function of the inventory level ($h_t(I_t)$). Notice that concave cost functions may include set-up costs. The notations are summarized in Table 1.

The CLSP-MOQ can be easily modeled by a mathematical program. The decision variables are X_t, I_t and a decision variable Y_t defined as follows:

$$Y_t = \begin{cases} 1 & \text{if the } X_t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The mathematical formulation of the CLSP-MOQ is then:

$$\text{Min} \sum_{t=1}^{T} p_t(X_t) + \sum_{t=1}^{T} h_t(I_t) \quad (1)$$

$$X_t + I_{t-1} - I_t = d_t \qquad \forall t \in T \quad (2)$$

$$LY_t \leqslant X_t \leqslant UY_t \qquad \forall t \in T \quad (3)$$

$$X_t, I_t \in \mathbb{R} \qquad \forall t \in T \quad (4)$$

$$Y_t \in \{0, 1\} \qquad \forall t \in T \quad (5)$$

The objective function (1) minimizes the total production and storage cost. Constraint (2) is the flow constraint. Constraint (3) insures that the maximum capacity and the minimum order quantity are satisfied. Constraints (4) and (5) define the validity domain of the variables.

Without loss of generality, we assume that $I_0 = 0$. Unfortunately, I_T can be strictly positive in an optimal solution. These two cases ($I_T = 0$ and $I_T > 0$) are considered in the following section.

Table 1
Notations.

T	Number of periods
d_t	Demand at period t
X_t	Production at period t
I_t	Inventory level at the end of period t
U	Production capacity
L	Minimum order quantity
$p_t(X_t)$	Concave production cost function
$h_t(I_t)$	Concave storage cost function

3. An optimal algorithm

In this section, we introduce some definitions and we prove some properties. Based on these properties, we will be able to derive a polynomial time algorithm to solve the CSLP-MOQ problem.

Definition 1 (*Regeneration points*). A period t is called a *regeneration point* if $I_t = 0$.

Definition 2 (*Fractional production periods*). A period t is called a *fractional production period* if $L < X_t < U$.

Definition 3 (*Sequence of production quantities*). The sequence of production quantities from $u + 1$ to v is noted S_{uv}.

Definition 4 (*UL-capacity-constrained sequences*). S_{uv} is a *UL*-capacity-constrained sequence if the following conditions are verified:

- u and v are regeneration points, i.e., $I_u = I_v = 0$;
- The demand d_t for $t = \{u + 1, \ldots, v\}$ is satisfied;
- For all $t \in \{u + 1, \ldots, v - 1\}$, $I_t \neq 0$, i.e., t is not a regeneration point;
- The production X_t for $t \in \{u + 1, \ldots, v\}$ is equal to 0, U or L, except for at most one period which can be a fractional production period.

At this time, we consider that $I_T = 0$. The case for which $I_T > 0$ will be considered at the end of this section.

Property 1. *A solution to the CLSP-MOQ problem can be seen as succession of sub-sequences such that both the starting period and the ending period are regeneration points.*

Proof. Assuming that $I_k = 0$ for some $k \in \{1 \cdots n - 1\}$. An optimal solution can be found by independently finding solutions to the problems for the first k periods and for the last $T - k$ periods. Consequently, a production plan can be seen as a sequence of consecutive periods in such a way that the stock is empty at the beginning and at the end of each sequence. □

The problem now is to know if the production plan of each sub-sequence is easy to compute. Fortunately, these sub-sequences have good properties that allow us to find the optimal production plan polynomially.

Property 2. *Let us consider an interval of periods [u, v] such that $I_u = I_v = 0$. UL-capacity-constrained sequences are dominant (i.e., at least one optimal solution is a UL-capacity-constrained sequence).*

Proof. To prove this result, we show that if a solution S_{uv} is not a *UL*-capacity-constrained sequence, it cannot be an extreme point of the polyhedron, and consequently is dominated by an other solution. In order to prove this result, we show that the solution S_{uv} is a convex combination of two other feasible solutions. Let us consider a solution S_{uv} such that $I_u = I_v = 0$, $I_t \neq 0$ for $t \in \{u + 1, \ldots, v - 1\}$ and in such a way that there exists at least two *fractional production periods* (i.e., i and j are such that $u + 1 \leqslant i < j \leqslant v$ and $L < X_i, X_j < U$). Consequently, we can relocate a small value of production between X_i and X_j as follows. Let us define ω as the biggest production quantity we can relocate keeping the solution feasible, and without changing other production levels. Then: