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Innovative Applications of O.R.

A Multi-Period Renewal equipment problem

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ABSTRACT

This paper looks at a Multi-Period Renewal equipment problem (MPR). It is inspired by a specific real-life situation where a set of hardware items is to be managed and their replacement dates determined, given a budget over a time horizon comprising a set of periods. The particular characteristic of this problem is the possibility of carrying forward any unused budget from one period to the next, which corresponds to the multi-periodicity aspect in the model. We begin with the industrial context and deduce the corresponding knapsack model that is the subject of this paper. Links to certain variants of the knapsack problem are next examined. We provide a study of complexity of the problem, for some of its special cases, and for its continuous relaxation. In particular, it is established that its continuous relaxation and a special case can be solved in (strongly) polynomial time, that three other special cases can be solved in pseudo-polynomial time, while the problem itself is strongly *NP*-hard when the number of periods is unbounded. Next, two heuristics are proposed for solving the MPR problem. Experimental results and comparisons with the Martello&Toth and Dantzig heuristics, adapted to our problem, are provided.

1. Introduction

The *knapsack problem* (KP) has a significant place in the study of integer programming models with binary variables. In the standard knapsack problem the quantity $\sum_{i\in N} p_i x_i$ is to be maximized subject to the constraint $\sum_{i\in N} w_i x_i \leqslant b$, where $x_i \in \{0,1\}$, $N = \{1,2,\ldots,n\}$, p_i is the *value* or *profit* of item i, w_i is the *weight* of item i and b is the knapsack *capacity*, all assumed to be non-negative. In addition to the standard problem, a number of different variants of the problem have been put forward and investigated by researchers over the last decades. This paper looks at one of these variants that we have called the Multi-Period Renewal equipment problem (MPR). It is inspired by a specific real-life situation where a set of hardware items is to be managed and their replacement dates determined, given a budget over a time horizon comprising a set of periods. The particular characteristic of this variant of the problem is the possibility of carrying forward any unused budget from one period to the next, corresponding to the multi-periodicity aspect in the statement of the problem. The contribution of this paper is a new knapsack model originating from a real industrial context, together with a complete theoretical examination of the problem and its relations to other knapsack problems, and a set of efficient heuristics for solving it.

The paper is organized as follows. In Section 2 we present the specific industrial context that gave rise to our problem and the corresponding mathematical model. Section 3 looks at links to other knapsack problems. In Section 4 we investigate the computational complexity of the MPR problem as well as some of its special cases and its continuous relaxation. We show in particular that the MPR problem is strongly *NP*-hard when the number of periods is unbounded and weakly *NP*-hard for the bounded case. Finally, in Section 5, we propose two new heuristics for the MPR problem and recall how two other well-known heuristics, that is to say the Dantzig and Martello&Toth heuristics, are also suitable for solving MPR. We provide a comparative experimental study of all these heuristics.

2. From an industrial problem to a theoretical model

In some countries it is usual for a city, town or municipality to contract certain public utilities (water supply, electricity, etc.) out to private companies, usually under concessions, leases or management contracts. Under these arrangements, the public entity delegates the provision of the service for a time horizon M typically ranging from 15 to 25 years, while the private entity remains under a contractual obligation to spend a given amount of money (B) on the maintenance and renewal of equipment. The company's maintenance strategy is based on continuous renewal so as to ensure continuity of service and to avoid problems with antiquated plant. For the application in hand (water supply network), equipments have lifetimes that range from 50 to 100 years which is largely greater than the considered time horizon. This implies that at most one replacement occurs over time horizon M. However, in practice more than one replacement could happen for

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one equipment during the time horizon due to unpredictable failures. These situations are handled by the daily maintenance process, while this paper deals only with the strategic maintenance process. In line with an internal budgeting policy the company allots an annual budget b_j to such expenditure, that is $\sum_{j\in M} b_j = B$. Given that the entire budget has to be used up, the company carries any unused budget at year j over to the following years. For each piece of equipment the replacement cost is assumed to be constant over the time horizon M, because in practice there is no reliable information of the variation of these costs over time. On the other hand, the profit attributable to the replacement, is calculated according to a formula based on such elements as the probability of failure, the expected lifetime of the equipment, its importance in the industrial process, etc. Hence, the profit change along with time and this change corresponds to a certain deterioration process [1]. Thus the related cost, profit and budget coefficients are assumed to be known with certainty, in contrast to conventional renewal theory which relies on probability theory, (see for instance Cox [2]). From this point of view the problem is a simplified deterministic version of conventional renewal problems. However, there is one particular property that increases the difficulty of problems, that is to say the property of multi-periodicity. More specifically, any decision made in some period j impacts those made in subsequent periods.

Before formulating the mathematical model of the MPR problem, let us give the notation used throughout the paper. Let N be a set of n pieces of equipment, and M a horizon of m periods.

- $x_{i,j}$ is the assignment decision variable, that is to say $x_{i,j} = 1$ if equipment i is replaced in period j, and 0 otherwise;
- $p_{i,j}$ is the *profit* obtained when replacing equipment i at period j;
- w_i is the *replacement cost* of equipment i (it remains unchanged over periods);
- b_j gives the *budget* alloted to period j.

All these data are assumed to be non-negative integers. Our problem can be mathematically formulated as follows:

$$\max \sum_{i\in N}\sum_{j\in M} p_{i,j}x_{i,j}$$
$$\sum_{i\in N} w_i x_{i,1} \leqslant b_1,$$
$$\sum_{i\in N} w_i x_{i,2} \leqslant b_2 + b_1 - \sum_{i\in N} w_i x_{i,1},$$
$$\dots$$
$$\sum_{i\in N} w_i x_{i,j} \leqslant b_j + \sum_{t=1}^{j-1} b_t - \sum_{t=1}^{j-1}\sum_{i\in N} w_i x_{i,t}, \quad \forall j \in M,$$
$$\sum_{j\in M} x_{i,j} \leqslant 1, \quad \forall i \in N,$$
$$x_{i,j} \in \{0,1\}, \quad \forall i \in N, \quad j \in M.$$

In the above formulation, the term $b_1 - \sum_{i\in N} w_i x_{i,1}$ gives the unused budget at the end of the first year. This is added to the allotted budget for the second year, and so on. Let B_j denote the cumulative budget from period 1 to period j, that is $B_j = \sum_{t=1}^{j} b_t$. Since all b_j are assumed to be non-negative, we have the following relation: $B_1 \leqslant B_2 \leqslant \cdots \leqslant B_m$. The problem can then be rewritten as follows:

$$(P) \qquad \max \sum_{j\in M}\sum_{i\in N} p_{i,j}x_{i,j} \tag{1}$$
$$\sum_{t=1}^{j}\sum_{i\in N} w_i x_{i,t} \leqslant B_j, \quad \forall j \in M, \tag{2}$$
$$\sum_{j\in M} x_{i,j} \leqslant 1, \quad \forall i \in N, \tag{3}$$
$$x_{i,j} \in \{0,1\}, \quad \forall i \in N, \quad j \in M. \tag{4}$$

Note that when $B_j = B_{j+1}$ for two consecutive periods j and $j + 1$, it can easily be shown that only the capacity constraint related to period $j + 1$ needs to be retained, and for each item the choice having the lower profit over these two periods may simply be discarded. Hence, we assume that $B_1 < B_2 < \cdots < B_m$.

From now on, in line with the notation commonly used for knapsack models, we shall use the term *item* for equipment, *weight* instead of replacement cost, and *capacity* for budget. Hence, item i assigned to period j reads equipment i replaced during period j. In the above model the sum of the weights of all items chosen from period 1 to period j cannot exceed capacity B_j for all $j \in M$ (2). Each item i can be assigned to at most one period j (3). The multi-period aspect lies in the fact that each constraint involves the current period and all preceding ones. The total profit is to be maximized (1). As far as we know, we were the first to model the problem in [3]. In the following we establish and exhibit links with some other problems studied in the literature.

3. Literature review and links with other knapsack problems

Let us first look at the *Multi-Period Knapsack problem* (MPK) introduced by Faaland [4]. Faaland considers a set N of items and a set M of periods. To each period $j \in M$ there corresponds a subset $N_j = \{j \in 1,\ldots,m\}$ of items that can be assigned to this period. Note that $\bigcup_{j=1}^{m} N_j = N$ and $N_k \cap N_j = \emptyset$ for each pair $(j,k)|j \neq k$ of items. For each item i, a *profit* p_i and a *weight* w_i are given. The cumulative weight of all items chosen from period 1 to period j cannot exceed the *capacity* B_j associated with period j. The total profit has to be maximized by selecting items in their associated periods. The decision variables in this problem are unbounded: an item i may be chosen more than once in period j such that $i \in N_j$. Faaland proposed a polynomial algorithm to solve exactly the continuous relaxation of MPK, and in so doing to compute an upper bound of MPK. He also proposed a *branch and bound* algorithm using this upper bound.

The **binary version** of this problem, in which an item is chosen at most once in its associated period, is called BMPK. Given the above notation, BMPK can be formulated as follows:

$$\max \sum_{j\in M}\sum_{i\in N_j} p_i x_i, \tag{5}$$

$$\sum_{t=1}^{j}\sum_{i\in N_t} w_i x_i \leqslant B_j, \quad \forall j \in M, \tag{6}$$

$$x_i \in \{0,1\}, \quad \forall i \in N. \tag{7}$$

Thus the weight of any chosen item will impact subsequent periods, given that in each period the cumulative weight is considered. The overall profit is maximized by choosing items in each period (5), without violating the cumulative capacity constraints (6). It now becomes apparent that BMPK and MPR have some similarities. What is different is that items in BMPK can be only assigned to a single (i.e. its associated) period. As we shall show, BMPK may be viewed as special case of MPR (see Section 4.3), in which an item may be chosen on at most one occasion. BMPK is shown to be weakly *NP*-hard in Section 4.3, and its complexity does not depend on the number of periods.

The *Generalized Assignment Problem* (GAP) is another problem related to knapsack problems. GAP is known to be strongly *NP*-hard and has been widely studied, (see for instance Martello