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Discrete Optimization

Tight bounds for periodicity theorems on the unbounded Knapsack problem

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ABSTRACT

Three new bounds for periodicity theorems on the unbounded Knapsack problem are developed. Periodicity theorems specify when it is optimal to pack one unit of the best item (the one with the highest profit-to-weight ratio). The successive applications of periodicity theorems can drastically reduce the size of the Knapsack problem under analysis, theoretical or empirical. We prove that each new bound is tight in the sense that no smaller bound exists under the given condition.

1. Introduction

The Knapsack problem is one of the most celebrated problems in operations research; not only because of its direct application to problems arising in the real world, but also because of its contribution to the solution methods for integer programming problems [2]. The unbounded Knapsack problem (UKP) can be stated as follows. Given a knapsack with known weight capacity and an unlimited supply of items, each with a given unit profit and unit weight, how can one pack the knapsack with integral amounts of items so as to maximize the profit of the load carried?

It is well known that this problem is NP-hard [3]. Many researchers have discovered numerous properties of the problem and developed a host of algorithms to utilize these findings [1,9–11,16,18,19]. A comprehensive discussion can be found in [6], and very recent efforts are presented in [13,17]. The two classic approaches for solving the Knapsack problem are branch and bound [10] and dynamic programming [12]. However, it is often possible to drastically reduce the size of the problem to be solved even before applying one of these approaches. Specifically, one such way of cutting down the computational requirements of problems with large, but bounded weight capacities is to employ turnpike theorems [4,8,15]. The turnpike theorem is also described as periodicity property [5]. If the items are indexed according to the non-increasing order of their profit-to-weight ratios, then for a large enough weight capacity it can be shown that it is optimal to pack at least one unit of the best item (the one with the highest profit-to-weight ratio). Periodicity theorems specify lower bounds on what constitute such large enough capacity, and their successive applications can drastically reduce the right-hand-sides. "Dynamic programming approaches are (often unjustly) rejected out of hand for large capacities, and it is thus important to study how the capacity affects the running time [1]." The periodicity property is a well known approach to reduce the search space for dynamic programming based algorithms.

The primary goal of this paper is to provide tight bounds and, thus, more effective periodicity theorems for the unbounded Knapsack problem. The plan of our paper is as follows. In Section 2, we introduce the notation and then develop the first tight bound, which is shown to subsume a known result attributed to Hu [8] by Garfinkel and Nemhauser [4]. In Section 3, we develop the second tight bound and show that it subsumes another known result from Garfinkel and Nemhauser [4]. The third tight bound, which is built on top of the first result, is developed in Section 4. Number theory is used to prove new theorems. Experimental results are discussed in Section 5 and the paper concludes in Section 6.

2. Preliminaries and the first periodicity theorem

The unbounded Knapsack problem (UKP) can be stated as follows: given an unlimited number of items of n types, where each item of type $j \in \{1,2,\dots,n\}$ has a weight $w_j > 0$, and a profit $p_j > 0$, how can one fill a knapsack with weight capacity $c > 0$ to maximize the profit of the load carried? More formally, one wants to find non-negative integers $\{x_1, x_2, \dots, x_n\}$ in order to

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$$\text{maximize } \sum_{j=1}^{n} p_j x_j$$
$$\text{subject to } \sum_{j=1}^{n} w_j x_j \leqslant c,$$

where all data involved are assumed integral. It is known that dynamic programming takes O(nc) time to solve a UKP. As noted in the introduction, one way to reduce the size of a problem with large, but bounded, right-hand-side is to employ periodicity theorems. If we index items according to the non-increasing order of their profit-to-weight ratios, $v_j = p_j/w_j$, so that $v_1 \geqslant v_2 \geqslant v_3 \geqslant \ldots$, then for large enough capacity we can prove that it is optimal to pack at least one unit of *item 1* (the item corresponding to the highest ratio v_1) into the knapsack. A periodicity theorem specifies a lower bound on what constitutes such large enough capacity under a given condition.

In constructing an optimal solution of a UKP, we (inductively) want to decide if at least one unit of *item 1* should be packed into the knapsack. Certainly if it is optimal to pack one unit of *item 1*, then there is a reduction of the (remaining) capacity. Before presenting our first theorem, we will state and prove a simple lemma.

Lemma 2.1. *Let q be a positive integer and s be a real number where $q \leqslant s$, then we have*

$$\frac{q}{q+1}s < \lfloor s \rfloor, \quad \text{where } \lfloor s \rfloor = \text{largest integer } \leqslant s.$$

Proof. [1] The left-hand-side of the inequality is increasing with integer q, and $q \leqslant s \Rightarrow q \leqslant \lfloor s \rfloor$. Therefore, it is sufficient to show its validity for $q = \lfloor s \rfloor$. Since for any real number s, we have $s < \lfloor s \rfloor + 1$. Clearly $\frac{\lfloor s \rfloor}{\lfloor s \rfloor + 1} s < \lfloor s \rfloor$ holds, and thus completes the proof. □

Note that the interval $[0, 1)$ can be written as the disjoint union of intervals of the form $\left[\frac{q-1}{q}, \frac{q}{q+1}\right)$, i.e., $[0,1) = \bigcup_{q=1}^{\infty} \left[\frac{q-1}{q}, \frac{q}{q+1}\right)$, and any real number r, $0 \leqslant r < 1$, falls precisely into one of the above subintervals. We now have the following theorem.

Theorem 2.2. *Assume $\frac{p_1}{w_1} > \frac{p_2}{w_2}$, i.e., $v_1 > v_2$, and let the positive integer q be uniquely determined by $\frac{q-1}{q} \leqslant \frac{v_2}{v_1} < \frac{q}{q+1}$, then there is a bound of capacity $h_I = qw_1$ such that if $c \geqslant h_I$, any optimal solution includes at least one unit of item 1.*

Proof. Given $v_1 > v_2 \geqslant v_3 \geqslant \ldots$, if none of *item 1* is packed, then the total value z of the objective function must have $z \leqslant cv_2$, it leads to $z < c\left(\frac{q}{q+1} v_1\right)$ due to the assumption that $\frac{v_2}{v_1} < \frac{q}{q+1}$. Therefore, we have

$$z < c\frac{q}{q+1} v_1 \tag{2.1}$$

and we want to show that any solution without *item 1* cannot be optimal under the given condition.

First, we claim that

$$c\frac{q}{q+1} v_1 < \left\lfloor \frac{c}{w_1} \right\rfloor p_1. \tag{2.2}$$

Note that the right-hand-side of (2.2) is the value of filling the knapsack with as many units of *item 1* as possible (which is certainly allowed). The above claim establishes our theorem.

Now let us prove our claim. By definition, $\frac{v_1}{p_1} = \frac{1}{w_1}$. Thus, inequality (2.2) is equivalent to $\left(\frac{q}{q+1}\right)\frac{c}{w_1} < \left\lfloor \frac{c}{w_1} \right\rfloor$, and this follows at once from Lemma 2.1 by taking $s = \frac{c}{w_1}$ (note that $c \geqslant qw_1$ in the assumption, so $q \leqslant \frac{c}{w_1}$ satisfies). Therefore, inequality (2.2) holds.

Inequalities (2.1) and (2.2) together yield $z < \left\lfloor \frac{c}{w_1} \right\rfloor p_1$.

The above right-hand-side is equal to the objective function value of filling the knapsack with allowable units of *item 1*. Thus, the initial solution without *item 1* cannot be optimal. Therefore, any optimal solution must include at least one unit of *item 1*. □

In fact, $h_I = qw_1$ is a tight bound in the sense that no smaller bound exists under the conditions specified in Theorem 2.2. The following proposition validates this.

Proposition 2.3. *Given the assumptions of Theorem 2.2, there exist examples such that if $c = h_I - 1$, then it is not optimal to pack item 1.*

Proof. Consider three types of items with $v_1 = 2q + 1$, $v_2 = 2q - 1$, and $v_3 = \frac{1}{2}$, where q is any positive integer. Their corresponding unit weights are $w_1 > 4q$, $w_2 = qw_1 - 1$, and $w_3 = 1$. It is easy to verify that $v_1 > v_2$, and $\frac{q-1}{q} \leqslant \frac{v_2}{v_1} < \frac{q}{q+1}$. Hence, we have a bound $h_I = qw_1$ by Theorem 2.2. Now let $c = h_I - 1 = qw_1 - 1$. Apparently, we can pack one unit of *item 2* into the knapsack, and the objective function value is

$$z^* = v_2 w_2 = (2q-1)(qw_1 - 1) = 2w_1 q^2 - w_1 q - 2q + 1.$$

We claim that for any solution including *item 1*, the resulting value is strictly less than z^*. Suppose we pack one unit of *item 1*, then the remaining capacity becomes $c - w_1 = (qw_1 - 1) - w_1 < qw_1 - 1 = w_2$. Thus, no *item 2* will fit and it limits our choices to *item 1* and *item 3* only. We assume that altogether there are t units of *item 1* packed (it must have $t < q$ since $c = qw_1 - 1$), and the rest capacity is filled with a suitable units of *item 3*. Thus, the resulting value z is

$$z = tw_1 v_1 + (c - tw_1) \cdot 1 \cdot v_3 = tw_1(2q+1) + (qw_1 - 1 - tw_1)\frac{1}{2}$$
$$= 2tw_1 q + \frac{1}{2}tw_1 + \frac{1}{2}qw_1 - \frac{1}{2}.$$

Notice that z increases with t, and it reaches its maximum when $t = q - 1$. Therefore, we have

$$z \leqslant 2(q-1)w_1 q + \frac{1}{2}(q-1)w_1 + \frac{1}{2}qw_1 - \frac{1}{2}$$
$$= 2w_1 q^2 - w_1 q - \frac{1}{2}(w_1 + 1) < 2w_1 q^2 - w_1 q - \frac{1}{2}(4q+1)$$
$$< 2w_1 q^2 - w_1 q - 2q + 1 = z^*.$$

Hence, it is not optimal to pack *item 1* into the knapsack. Thus, one cannot improve on h_I. □

We also observe that Theorem 2.2 subsumes a known result, attributed to Hu [8] by Garfinkel and Nemhauser [4]. Hu's result is less emphatic since it states only the existence of an optimal solution that includes *item 1*.

Corollary 2.4. [Hu's bound] *Let us assume that $\frac{p_1}{w_1} > \frac{p_2}{w_2}$, i.e., $v_1 > v_2$, then there is a bound of capacity $k_I = \frac{p_1}{v_1 - v_2}$ such that if $c \geqslant k_I$, it is optimal to pack at least one unit of item 1.*

Proof. Refer to notation in Theorem 2.2. There exists a unique positive integer q that is the maximal possible and satisfies the following inequality,

$$\frac{q-1}{q} \leqslant \frac{v_2}{v_1} \iff v_1 - \frac{1}{q}v_1 \leqslant v_2 \iff v_1 - v_2 \leqslant \frac{1}{q}v_1 \iff$$
$$\frac{p_1}{v_1 - v_2} \geqslant q\left(\frac{p_1}{v_1}\right) = qw_1.$$

i.e., $k_I \geqslant h_I$. Thus, $c \geqslant k_I$ implies $c \geqslant h_I$, and the corollary follows from Theorem 2.2. □

[1] This proof was suggested by one of the referees.