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Continuous Optimization

A specialized network simplex algorithm for the constrained maximum flow problem

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ABSTRACT

The constrained maximum flow problem is to send the maximum flow from a source to a sink in a directed capacitated network where each arc has a cost and the total cost of the flow cannot exceed a budget. This problem is similar to some variants of classical problems such as the constrained shortest path problem, constrained transportation problem, or constrained assignment problem, all of which have important applications in practice. The constrained maximum flow problem itself has important applications, such as in logistics, telecommunications and computer networks. In this research, we present an efficient specialized network simplex algorithm that significantly outperforms the two widely used LP solvers: CPLEX and lp_solve. We report CPU times of an average of 27 times faster than CPLEX (with its dual simplex algorithm), the closest competitor of our algorithm.

1. Introduction

Let $G = (N,A)$ be a directed network consisting of a set N of nodes and a set A of arcs. In this network, the flow on each arc (i,j) is represented by the nonnegative variable x_{ij}, with a cost c_{ij} and capacity u_{ij}. The constrained maximum flow problem is to send the maximum possible flow from a source node s to a sink node t where the total cost of the flow is constrained by the budget, D. For ease of exposition, we assume that there is an arc $(t,s) \in A$ with $c_{ts} = 0$ and $u_{ts} = \infty$, and that there exists a path from s to t in the network. The problem then can be formulated as the following linear program:

$$[\text{CMF\_LP}] \quad \max \quad x_{ts} \tag{1.1a}$$

s.t.

$$\sum_{(i,j)\in A} x_{ij} - \sum_{(j,i)\in A} x_{ji} = 0 \quad \forall i \in N, \tag{1.1b}$$

$$\sum_{(i,j)\in A} c_{ij}x_{ij} \leqslant D, \tag{1.1c}$$

$$0 \leqslant x_{ij} \leqslant u_{ij} \quad \forall (i,j) \in A. \tag{1.1d}$$

Eq. (1.1b) are called the *balance* or *conservation of flow* constraints. Eq. (1.1c) is called the *budget* constraint. Eq. (1.1d) are the *upper* and *lower bound* constraints.

It is important to study this problem both from theoretical and practical points of view, because it may provide insight into similar important problems such as the constrained shortest path problem, constrained transportation problem, or constrained assignment problem, all of which have important applications in practice. The constrained maximum flow problem itself has important applications as well. Some examples can be described as follows: (1) In a physical distribution network, each node represents a distribution center, which could be an origin, destination, or a transit point for the flow of goods. Each arc represents a mode of transportation between two distribution centers and has a cost and capacity. The objective is to determine the capacity of the distribution network, or the maximum flow between all origin and destination nodes subject to the transportation budget. (2) The same framework can be used to model computer networks where the objective is to maximize the packets transferred between sources and sinks and the total transfer cost is constrained.

In this research, we present an efficient specialized network simplex algorithm for the constrained maximum flow problem, and compare its computational performance against the two widely used LP solvers CPLEX and lp_solve, as well as the double scaling algorithm that we propose in Çalışkan (2008). We used two network generators in our experiments: the generator that we describe in Çalışkan (2008) and the NETGEN generator that is described in Klingman et al. (1974). The key contribution of our research is that the specialized simplex algorithm is directly carried out on the network in a particularly efficient fashion. The algorithmic steps of the simplex algorithm (finding improving columns, updating the basis components, and re-computing the primal and dual solutions) are carried out efficiently by isolating and exploiting the maximum flow network substructure and integrating

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the side constraint into the tree structure of the basis. To our knowledge, our algorithm is the first to specialize the simplex algorithm for the constrained maximum flow problem.

The rest of the paper is organized as follows: first, we provide a review of the related literature in Section 2; then we present the details of the proposed algorithm and discuss an extension to the algorithm in Section 3; then we provide the results of our computational experiments in Section 4; and finally, we present the conclusions in Section 5.

2. Literature review

Network flow problems with side constraints may sometimes be solved as pure network problems by replacing the side constraints with a number of flow balance equations and a few additional nodes and arcs (Glover et al., 1974; Klingman, 1977). Unfortunately, this transformation is not possible in general, but significant computational improvement is still possible by specializing the simplex algorithm. Klingman and Russell (1975, 1978), Glover et al. (1978) propose specializations for the constrained transportation and transshipment problems. Graves and McBride (1976) propose a general framework for the simplex algorithm where an embedded network structure, generalized upper bound constraints, and block diagonal structure are present. Brown and Olson (1994) present a general row factorization framework for generalized upper bound and network rows. Chen and Saigal (1977), Glover and Klingman (1981, 1985) propose specialized simplex algorithms for networks with side constraints. McBride (1985) proposes another specialized simplex algorithm called EMNET that handles networks with side constraints and side variables. McBride and Mamer (1997, 2001), McBride (1998), Mamer and McBride (2000) describe several implementations of the EMNET algorithm for the multicommodity flow problem. Spälti and Liebling (1991) study a special case of the singly constrained network flow problem that arises in the context of an optimal satellite placement problem and propose a specialized simplex algorithm. More recently, Fang and Qi (2003), Mo et al. (2005a,b), Lu et al. (2006), Venkateshan et al. (2008) specialize the generalized network simplex algorithm for networks that arise in the context of assembly and distillation operations in supply chains.

Lagrangian relaxation is also used for networks with side constraints. Belling-Seib et al. (1988) compares Lagrangian relaxation with primal and dual simplex for the singly constrained network flow problem, whereas Bryson (1991) proposes another Lagrangian relaxation approach. Mathies and Mevert (1998) propose a hybrid of primal simplex and Lagrangian relaxation.

Since the labeling algorithm of Ford and Fulkerson (1956), many algorithms were proposed for the classical maximum flow problem, but its variants received little interest. Fulkerson (1959) describes a maximum flow problem with an additional convex budget constraint. Malek-Zavarei and Frisch (1971) describe a maximum flow problem with side constraints on some nodes and propose a heuristic solution. Ahuja and Orlin (1995) introduce the constrained maximum flow problem that we study in this paper and propose a polynomial time capacity scaling algorithm. In Çalışkan (2009), we point out that in some cases Ahuja and Orlin (1995)'s algorithm does not converge to the optimal solution and propose a slight modification to the algorithm. To this date, no specialized simplex algorithm was proposed for the constrained maximum flow problem. In the simplex algorithm specialized for the maximum flow problem, dual variables and reduced costs do not require explicit calculation due to the special structure of the basis. If the simplex algorithm is specialized for the constrained maximum flow problem, the resulting algorithm will benefit from this special structure as well. To our knowledge, our paper is the first to exploit the special structure of the maximum flow problem to devise an efficient simplex implementation for the constrained maximum flow problem.

Throughout the next section, we present the theoretical development of the specialized simplex algorithm for the constrained maximum flow problem. We only present what is new in our theoretical development and give appropriate references to prior work whenever it is necessary.

3. The specialized network simplex algorithm

A basis of the maximum flow problem consists of a pair of subtrees that are rooted at nodes s and t connected by the arc (t,s), which is always basic. The two subtrees and the artificial arc form a spanning tree of the network (see Ahuja et al., 1993, p. 430). We denote the spanning tree T; the set of arcs in the source subtree, S; and the set of arcs in the sink subtree, Z. In the constrained maximum flow problem, there is one more arc in the basis that forms a unique cycle on the spanning tree, which is called a *fundamental cycle*. We denote the additional arc, (a,b); its fundamental cycle, $\psi(a,b)$; and the total (net) cost around $\psi(a,b)$ when the cycle is traversed in the direction of (a,b), $C_{\psi(a,b)}$. The arc (a,b) connects either two nodes from different subtrees, or two nodes of the same subtree. We call the former an "inter-subtree arc," and the latter an "intra-subtree arc" (see Fig. 1). Thus, a basis of the constrained maximum flow problem consists of the subtrees S and Z, the non-tree basic arc (a,b), and the arc (t,s).

Lemma 1. *For a basis B of the constrained maximum flow problem,* $C_{\psi(a,b)} \neq 0$.

Proof. Assume that $C_{\psi(a,b)} = 0$. If we add all columns of [CMF_LP] corresponding to the arcs of $\psi(a,b)$, the result will be 0 for all rows corresponding to the balance equations and $C_{\psi(a,b)}$ for the row corresponding to the budget constraint. But this means these arcs are not linearly independent and B cannot be a basis. Therefore, $C_{\psi(a,b)}$ must be nonzero. □

The proposed specialized simplex algorithm solves a maximum flow problem. If the maximum flow from s to t does not violate the budget constraint, then this maximum flow is the optimal solution to the constrained maximum flow problem. Otherwise, there is an optimal solution that satisfies the budget constraint exactly. Henceforth, we assume that the budget constraint is satisfied exactly.

3.1. Optimality conditions

We denote the dual variables corresponding to the balance equations by $\boldsymbol{\pi}$, and the dual variable corresponding to the budget constraint by λ. We define the reduced cost for an arc $(i,j) \in A$ as $\bar{c}_{ij} = -\pi_i + \pi_j - \lambda c_{ij}$. For a given flow $\boldsymbol{x}$, we denote the set of arcs with flow equal to their upper bounds by U, and the set of arcs with flow equal to their lower bounds by L. Then, $\boldsymbol{x}$ will be optimal if and only if:

$$\bar{c}_{ij} \leqslant 0 \text{ for every arc } (i,j) \in L,$$
$$\bar{c}_{ij} \geqslant 0 \text{ for every arc } (i,j) \in U. \tag{3.1}$$

If an arc violates the above optimality conditions, then we may enter it into the basis and improve the solution.

Lemma 2. *If* (a,b) *is an inter-subtree arc from S to Z,* $\lambda = 1/C_{\psi(a,b)}$; *if it is an inter-subtree arc from Z to S,* $\lambda = -1/C_{\psi(a,b)}$; *and if it is an intra-subtree arc,* $\lambda = 0$.