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Interfaces with Other Disciplines

Robust portfolio optimization with derivative insurance guarantees

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ABSTRACT

Robust portfolio optimization aims to maximize the worst-case portfolio return given that the asset returns are allowed to vary within a prescribed uncertainty set. If the uncertainty set is not too large, the resulting portfolio performs well under normal market conditions. However, its performance may substantially degrade in the presence of market crashes, that is, if the asset returns materialize far outside of the uncertainty set. We propose a novel robust optimization model for designing portfolios that include European-style options. This model trades off weak and strong guarantees on the worst-case portfolio return. The weak guarantee applies as long as the asset returns are realized within the prescribed uncertainty set, while the strong guarantee applies for all possible asset returns. The resulting model constitutes a convex second-order cone program, which is amenable to efficient numerical solution procedures. We evaluate the model using simulated and empirical backtests and analyze the impact of the insurance guarantees on the portfolio performance.

1. Introduction

Investors face the challenging problem of how to distribute their current wealth over a set of available assets, such as stocks, bonds, and derivatives, with the goal to earn the highest possible future wealth. One of the first mathematical models for this problem was formulated by Markowitz [32]. In his Nobel prize-winning work, he observed that a rational investor does not aim solely at maximizing the expected return of an investment, but also at minimizing its risk. In the Markowitz model, the risk of a portfolio is measured by the variance of the portfolio return. A practical advantage of the Markowitz model is that it reduces to a convex quadratic program, which can be solved efficiently.

Although the Markowitz model has triggered a tremendous amount of research activities in the field of finance, it has serious disadvantages which have discouraged practitioners from using it. The main problem is that the means and covariances of the asset returns, which are important inputs to the model, have to be estimated from noisy data. Hence, these estimates are not accurate. In fact, it is fundamentally impossible to estimate the mean returns with statistical methods to within workable precision, a phenomenon which is sometimes referred to as *mean blur* [29,34]. Unfortunately, the mean–variance model is very sensitive to the distributional input parameters. As a result, the model amplifies any estimation errors, yielding extreme portfolios which perform badly in out-of-sample tests [16,12,36,18].

Many attempts have been undertaken to ease this amplification of estimation errors. Black and Litterman [10] suggest Bayesian estimation of the means and covariances using the market portfolio as a prior. Jagannathan and Ma [26] as well as Chopra [14] impose portfolio constraints in order to guide the optimization process towards more intuitive and diversified portfolios. Chopra et al. [15] use a James–Steiner estimator for the means which tilts the optimal allocations towards the minimum-variance portfolio, while DeMiguel and Nogales [18] employ robust estimators.

In recent years, *robust optimization* has received considerable attention. Robust optimization is a powerful modeling paradigm for decision problems subject to non-stochastic data uncertainty [6]. The uncertain problem parameters are assumed to be unknown but confined to an *uncertainty set*, which reflects the decision maker's uncertainty about the parameters. Robust optimization models aim to find the best decision in view of the worst-case parameter values within these sets. Ben-Tal and Nemirovski [7] propose a robust optimization model to immunize a portfolio against the uncertainty in the asset returns. They show that when the asset returns can vary within an ellipsoidal uncertainty set determined through their means and covariances, the resulting optimization problem is reminiscent of the Markowitz model. This robust portfolio selection model still assumes that the distributional input parameters are known precisely. Therefore, it suffers from the same shortcomings as the Markowitz model.

Robust portfolio optimization can also be used to immunize a portfolio against the uncertainty in the distributional input parameters. Goldfarb and Iyengar [22] use statistical methods for constructing uncertainty sets for factor models of the asset returns

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and show that their robust portfolio problem can be reformulated as a second-order cone program. Tütüncü and Koenig [41] propose a model with box uncertainty sets for the means and covariances and show that the arising model can be reduced to a smooth saddle-point problem subject to semidefinite constraints. Rustem and Howe [39] describe algorithms to solve general continuous and discrete minimax problems and present several applications of worst-case optimization for risk management. Rustem et al. [38] propose a model that optimizes the worst-case portfolio return under rival risk and return forecasts in a discrete minimax setting. El Ghaoui et al. [20] show that the worst-case Value-at-Risk under partial information on the moments can be formulated as a semidefinite program. Ben-Tal et al. [5] as well as Bertsimas and Pachamanova [9] suggest robust portfolio models in a multi-period setting. A recent survey of applications of robust portfolio optimization is provided in the monograph [21]. Robust portfolios of this kind are relatively insensitive to the distributional input parameters and typically outperform classical Markowitz portfolios [13].

Robust portfolios exhibit a *non-inferiority* property [38]: whenever the asset returns are realized within the prescribed uncertainty set, the realized portfolio return will be greater than or equal to the calculated worst-case portfolio return. Note that this property may fail to hold when the asset returns happen to fall outside of the uncertainty set. In this sense, the non-inferiority property only offers a *weak guarantee*. When a rare event (such as a market crash) occurs, the asset returns can materialize far beyond the uncertainty set, and hence the robust portfolio will remain unprotected. A straightforward way to overcome this problem is to enlarge the uncertainty set to cover also the most extreme events. However, this can lead to robust portfolios that are too conservative and perform poorly under normal market conditions.

In this paper we will use portfolio insurance to hedge against rare events which are not captured by a reasonably sized uncertainty set. Classical portfolio insurance is a well studied topic in finance. The idea is to enrich a portfolio with specific derivative products in order to obtain a deterministic lower bound on the portfolio return. The insurance holds for all possible realizations of the asset returns and can therefore be qualified as a *strong guarantee*. Numerous studies have investigated the integration of options in portfolio optimization models. Ahn et al. [1] minimize the Value-at-Risk of a portfolio consisting of a single stock and a put option by controlling the portfolio weights and the option strike price. Dert and Oldenkamp [19] propose a model that maximizes the expected return of a portfolio consisting of a single index stock and several European options while guaranteeing a maximum loss. Howe et al. [24] introduce a risk management strategy for the writer of a European call option based on minimax using box uncertainty. Lutgens et al. [30] propose a robust optimization model for option hedging using ellipsoidal uncertainty sets. They formulate their model as a second-order cone program which may have, in the worst-case, an exponential number of conic constraints.

This paper combines robust portfolio optimization and classical portfolio insurance with the objective of providing two layers of guarantees. The weak non-inferiority guarantee applies as long as the returns are realized within the uncertainty set, while the strong portfolio insurance guarantee also covers cases in which the returns are realized outside of the uncertainty set. The ideas set out in this paper are related to the concept of Comprehensive Robustness proposed by Ben-Tal et al. [4]. Comprehensive Robustness aims to control the deterioration in performance when the uncertainties materialize outside of the uncertainty set. Our work establishes the relationship between offering guarantees beyond the uncertainty set and portfolio insurance. Indeed, we will show that in order to control the deterioration in portfolio return, our model will allocate wealth in put and call options. The premia of these options will determine the cost to satisfy the guarantee levels. Our contributions can be summarized as follows:

(1) We extend the existing robust portfolio optimization models to include options as well as stocks. Because option returns are convex piece-wise linear functions of the underlying stock returns, options cannot be treated as additional stocks, and the use of an ellipsoidal uncertainty set is no longer adequate. Under a no short-sales restriction on the options, we demonstrate how our model can be reformulated as a convex second-order cone program that scales gracefully with the number of stocks and options. We also show that our model implicitly minimizes a *coherent risk measure* [3]. Coherency is a desirable property from a risk management viewpoint.

(2) We describe how the options in the portfolio can be used to obtain additional strong guarantees on the worst-case portfolio return even when the stock returns are realized outside of the uncertainty set. We show that the arising *Insured Robust Portfolio Optimization* model trades off the guarantees provided through the non-inferiority property and the derivative insurance strategy. Using conic duality, we reformulate this model as a tractable second-order cone program.

(3) We perform a variety of numerical experiments using simulated as well as real market data. In our simulated tests we illustrate the tradeoff between the non-inferiority guarantee and the strong insurance guarantee. We also evaluate the performance of the Insured Robust Portfolio Optimization model under "normal" market conditions, in which the asset prices are governed by geometric Brownian motions, as well as in a market environment in which the prices experience significant downward jumps. The impact of the insurance guarantees on the portfolio performance is also analyzed using real market prices.

The rest of the paper is organized as follows. In Section 2 we review robust portfolio optimization and elaborate on the non-inferiority guarantee. In Section 3 we show how a portfolio that contains options can be modelled in a robust optimization framework and how strong insurance guarantees can be imposed on the worst-case portfolio return. We also demonstrate how the resulting model can be formulated as a tractable second-order cone program. In Section 4 we report on numerical tests in which we compare the insured robust model with the standard robust model as well as the classical mean–variance model. We run simulated as well as empirical backtests. Conclusions are drawn in Section 5, and a notational reference table is provided in Appendix A.1.

2. Robust portfolio optimization

Consider a market consisting of n stocks. Moreover, denote the current time as $t = 0$ and the end of investment horizon as $t = T$. A portfolio is completely characterized by a vector of weights $\boldsymbol{w} \in \mathbb{R}^n$, whose elements add up to 1. The component w_i denotes the percentage of total wealth which is invested in the ith stock at time $t = 0$. Furthermore, let $\tilde{\boldsymbol{r}}$ denote the random vector of total stock returns over the investment horizon, which takes values in $\mathbb{R}^n_+$. By definition, the investor will receive $\tilde{r}_i$ dollars at time T for every dollar invested in stock i at time 0. We will always denote random variables by symbols with tildes, while their realizations are denoted by the same symbols without tildes.

The return vector $\tilde{\boldsymbol{r}}$ is representable as:

$$\tilde{\boldsymbol{r}} = \boldsymbol{\mu} + \tilde{\boldsymbol{\epsilon}}, \tag{1}$$