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# The axiomatic approach to three values in games with coalition structure ☆

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ABSTRACT

We present a unified framework for a broad class of values in transferable utility games with coalition structure, including the Owen coalitional value and two weighted versions with weights given by the size of the coalitions. We provide three axiomatic characterizations using the properties of Efficiency, Linearity, Independence of Null Coalitions, and Coordination, with two versions of Balanced Contributions inside a Coalition and Weighted Sharing in Unanimity Games, respectively.



## 1. Introduction

Coalition structures are important in many real-world contexts, such as the formation of cartels or bidding rings, alliances or trading blocs among nation states, research joint ventures, and political parties.

These situations can be modelled through transferable utility (*TU*, for short) games, in which the players partition themselves into coalitions for the purpose of bargaining. All players in the same coalition agree before the play that any cooperation with other players will only by carried out collectively. That is, either all the members of the coalition take part of it or none of them (Malawski, 2004).

Given a coalition structure, bargaining occurs between coalitions and between players in the same coalition. The main idea is that the coalitions play among themselves as individual players in a game among coalitions, and then, the profit obtained by each coalition is distributed among its members. Owen (1977) studied the allocation that arises from applying the Shapley value (Shapley, 1953b) twice: first in the game among coalitions, and then in a reduced game inside each coalition.

The same two-step approach has been applied by Casas-Méndez et al. (2003) to generate the $\tau$-value (Tijs, 1981), and by Pulido and Sánchez-Soriano (2009) to generalize the core and Weber set. Other generalization of the core is provided by Pulido and Sánchez-Soriano (2006).

In general, the two-step approach assumes a symmetric treatment for each coalition. As Harsanyi (1977) points out, in unanimity games this procedure implies that players would be better off bargaining by themselves than joining forces. This is known as the join-bargaining paradox, or the Harsanyi paradox.

An alternative approach is to give a different treatment, or weight, to each coalition. Following this idea, Levy and McLean (1989) apply the weighted Shapley value (Shapley, 1953a; Kalai and Samet, 1987, 1988) in the game among coalitions, as well as in the reduced games. Other weighted version of the Shapley value is provided by Haeringer (2006), whereas a weighted version of the Banzhaf value is provided by Radzik et al. (1997) and Nowak and Radzik (2000).

In all these works, the weight system is exogenously given. However, a natural weight for each coalition can also be endogenously provided by its own cardinality. In fact, a motivation for the weighted Shapley value is precisely the difference in size.[1] Moreover, Kalai and Samet (1987, Corollary 2 in Section 7) show that the cardinality of coalitions are appropriate weights for the players. The reason is that if we force the players in a coalition to work together (by destroying their resources when they are not all together), then the aggregated

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[1] Kalai and Samet (1987) present the example of large constituencies with many individuals, in contrast with constituencies composed by a small number of individuals.

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Shapley value of each coalition in the new game coincides with the weighted Shapley value of the game among coalitions, with weights given by the cardinality of the coalition.[2]

It is then reasonable to apply the Levy and McLean value with intracoalitional symmetry and weights given by the cardinality of the coalition. However, in Levy and McLean's model, the weight of the subcoalitions in the reduced game remains constant, even though these subcoalitions may have different cardinality. An alternative approach is to vary the weight of the coalitions in the reduced game. Vidal-Puga (2006) follows this approach to define a new coalitional value. This new coalitional value does not present the Harsanyi paradox.

We have then, three reasonable generalizations of the Shapley value for games with coalition structure: the coalitional Owen value (Owen, 1977), the coalitional Levy–McLean weighted value (Levy and McLean, 1989) with the weights given by the size of the coalition, and the new value presented by Vidal-Puga (2006). In order to compare the three coalitional values, we can study which properties are satisfied by each of them, so that we can decide from these properties which is the most reasonable one for each particular situation. Moreover, when these properties completely characterize the values, we can be sure that the properties catch their essence.

In this paper, we characterize the above coalitional values. These three values have in common the following feature: first, the worth of the grand coalition is divided among the coalitions following either the Shapley value (Owen), or the weighted Shapley value with weights given by the size of the coalitions (Levy and McLean, Vidal-Puga), and then the profit obtained by each coalition is distributed among its members following the Shapley value of an appropriately defined "reduced" game.

Some of the axioms used in the characterizations (*efficiency, intracoalitional symmetry*, and *linearity*) are standard in the literature, others (*independence of null coalitions* and two intracoalitional versions of *balanced contributions*) are used in many different frameworks. The property of independence of null coalitions is related to the standard property of *null player*. However, the role of null players is important as their only presence affects the size of the coalition. In fact, two of the three values (Owen, Levy and McLean) satisfy the null player property, whereas the third one (Vidal-Puga) does not.

Additionally, we introduce new properties in this kind of problems: *coordination* (which asserts that internal changes in a coalition which do no affect the game among coalitions, do not influence the final payment of the rest of the players) and two properties of *sharing in unanimity games* (which establish how should the payment be under the grand coalition in unanimity game).

The properties of efficiency, linearity, intracoalitional symmetry and independence of null coalitions are natural extensions of the classical properties that characterize the Shapley value (efficiency, linearity, symmetry and null player, respectively) to the game among coalitions. On the other hand, the properties of balanced contributions are applied to the game inside a coalition, and each of them is a natural extension of the property of balanced contribution that also characterizes, with efficiency, the Shapley value (Myerson, 1980). Moreover, the property of *balanced intracoalitional contributions* is used to characterize the value proposed by Vidal-Puga (2006) in Gómez-Rúa and Vidal-Puga (forthcoming). Hence, the three values proposed here can be seen as natural extensions of the Shapley value for games with coalition structure. Additionally, the property of coordination formalizes the idea presented by Owen that the players inside a coalition negotiate among them, but always assuming that the rest of the coalitions remain together (see for example the game $v_1$ defined by Kalai and Samet, 1987, Section 7).

The paper is organized as follows. In Section 2, we introduce the model. In Section 3, we define a family that includes the three coalitional values. In Section 4, we present the properties used in the characterization and we study which properties the coalitional values satisfy. In Section 5, we present the characterization results. In Section 6, we prove that the properties are independent. In Section 7, we present some concluding remarks.

## 2. Notation

Let $U = \{1, 2, \ldots\}$ be the (infinite) set of potential players.

Given a finite subset $N \subset U$, let $\Pi(N)$ denote the set of all orders in $N$. Given $\pi \in \Pi(N)$, let $Pre(i, \pi)$ denote the set of the elements in $N$ which come before $i$ in the order given by $\pi$, i.e. $Pre(i, \pi) = \{j \in N : \pi(j) < \pi(i)\}$. For any $S \subset N$, $\pi_S$ denotes the order induced in $S$ by $\pi$ (for all $i, j \in S$, $\pi_S(i) < \pi_S(j)$ if and only if $\pi(i) < \pi(j)$).

Let $|C|$ denote the cardinality of a set $C$.

A *transfer utility game*, *TU game*, or simply a *game*, is a pair $(N, v)$ where $N \subset U$ is finite and $v : 2^N \to \mathbb{R}$ satisfies $v(\emptyset) = 0$. When $N$ is clear, we can also denote $(N, v)$ as $v$. Given a *TU* game $(N, v)$ and $S \subset N$, $v(S)$ is called the *worth* of $S$. Given $S \subset N$, we denote the restriction of $(N, v)$ to $S$ as $(S, v)$.

For simplicity, we usually write $S \cup i$ instead of $S \cup \{i\}$, $N \setminus i$ instead of $N \setminus \{i\}$, and $v(i)$ instead of $v(\{i\})$.

Two players $i, j \in N$ are *symmetric* in $(N, v)$ if $v(S \cup i) = v(S \cup j)$ for all $S \subset N \setminus \{i, j\}$. A player $i \in N$ is *null* in $(N, v)$ if $v(T \cup i) = v(T)$ for all $T \subset N \setminus i$. The set of non-null players in $(N, v)$ is the *minimal carrier* of $(N, v)$, and we denote it as $MC(N, v)$. Given two games $(N, v), (N, w)$, the game $(N, v + w)$ is defined as $(v + w)(S) = v(S) + w(S)$ for all $S \subset N$. Given a game $(N, v)$ and a real number $\alpha$, the game $(N, \alpha v)$ is defined as $(\alpha v)(S) = \alpha v(S)$ for all $S \subset N$.

Given $N \subset U$ finite, we call *coalition structure* over $N$ a partition of the player set $N$, i.e. $\mathcal{C} = \{C_1, C_2, \ldots, C_m\} \subset 2^N$ is a coalition structure if it satisfies $\bigcup_{C_q \in \mathcal{C}} C_q = N$ and $C_q \cap C_r = \emptyset$ when $q \neq r$. We also assume $C_q \neq \emptyset$ for all $q$.

We say that $C_q \in \mathcal{C}$ is a *null coalition* if all its members are null players.

For any $S \subset N$, we denote the restriction of $\mathcal{C}$ to the players in $S$ as $\mathcal{C}_S$, i.e. $\mathcal{C}_S = \{C_q \cap S : C_q \in \mathcal{C} \text{ and } C_q \cap S \neq \emptyset\}$.

For any $S \subset C_q \in \mathcal{C}$, we will frequently study the case in which the players in $C_q \setminus S$ leave the game. In this case, we write $\mathcal{C}^S$ instead of the more cumbersome $\mathcal{C}_{N \setminus (C_q \setminus S)}$.

Given a game $(N, v)$ and a coalition structure $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ over $N$, the *game among coalitions* is the TU game $(M, v/\mathcal{C})$ where $M = \{1, 2, \ldots, m\}$ and $(v/\mathcal{C})(Q) = v\left(\bigcup_{q \in Q} C_q\right)$ for all $Q \subset M$.

We denote the *game* $(N, v)$ *with coalition structure* $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ over $N$ as $(N, v, \mathcal{C})$ or $(v, \mathcal{C})$.

[2] Another possibility is to give the worth of any coalition to any of its nonempty subcoalitions. In this case, the aggregated Shapley value of each coalition coincides with the weighted Shapley value of the *dual* game among coalitions (see Kalai and Samet, 1987, Section 7, for further details).