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Optimal asset allocation for aggregated defined benefit pension funds with stochastic interest rates ☆

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ABSTRACT

In this paper we study the optimal management of an aggregated pension fund of defined benefit type, in the presence of a stochastic interest rate. We suppose that the sponsor can invest in a savings account, in a risky stock and in a bond with the aim of minimizing deviations of the unfunded actuarial liability from zero along a finite time horizon. We solve the problem by means of optimal stochastic control techniques and analyze the influence on the optimal solution of some of the parameters involved in the model.

1. Introduction

Pension funds currently represent one of the most important institutions in financial markets because of their high investment capacity and because they complement the role of the Government, allowing those workers who have reached retirement age to maintain their standard of living. These two aspects justify the interest generated over recent years in the study of the optimum management of pension plans.

There are two principal alternatives in pension plan designs with respect to the assignment of risk. In a defined contribution (DC) plan the risk derived from the fund management is borne by the beneficiary. However, in a defined benefit (DB) plan, where the benefits are normally related to the final salary level, the financial risk is assumed by the sponsor agent.

Our aim in this paper is to analyze a BD pension fund of aggregated type, which is a common model in the employment system. We provide here an extension of the previous work of the authors, Josa-Fombellida and Rincón-Zapatero (2001, 2004, 2006, 2008a,b), in an attempt to incorporate more realistic assumptions to the model, dropping the hypothesis of a constant riskless rate of interest. Thus, in our model, there are three sources of uncertainty: (i) the fund assets returns; (ii) the instantaneous riskless rate of interest; and (iii) the evolution of benefits, based on the behavior of salaries and/or other main components of the pension plan.

There are several previous papers dealing with the management of DC funds in the presence of a stochastic rate of interest. Some of them are Boulier et al. (2001), Battocchio and Menoncin (2004), Cairns et al. (2006) and Menoncin (2005), where the interest rate is assumed to be of the Vasicek type. In Deelstra et al. (2003), the interest rate has an affine structure, as in Duffie and Kan (1996) which includes as a special case the CIR and the Vasicek models. Other interesting papers where the interest rate is random, though in a discrete time are Vigna and Haberman (2001) and Haberman and Vigna (2002). The importance of DB funds calls for the completion of the theory studying this case. Moreover, the differences in both types of pension plans makes it impossible to transfer the results from DC to DB plans.

The objective of the shareholder in a DC pension fund is to maximize the expected utility obtained from fund accumulation at a fixed date. The contribution rate is exogenous to this optimization process, since it is generally determined by salary. However, in a DB plan the amortization effort is a control variable. The fund assets could be artificially increased with high contributions. Obviously, this makes no sense, since benefits are fixed in advance. Thus, the objective in a DB plan should be related with risk minimization instead of the maximization of fund assets. Of course, the main concern of the sponsor is the solvency risk, related to the security of the pension fund in attaining the comprised liabilities. Similar objectives have been considered in other works, such as Haberman and Sung (1994),

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Haberman et al. (2000) and Josa-Fombellida and Rincón-Zapatero (2001, 2004). The optimal management of DB plans in the presence of a random interest rate is found, but in discrete time, in Haberman and Sung (1994), Chang (1999) and Chang et al. (2003).

We make the contribution rate endogenous and dependent on the main variables of the fund, by adopting a spread method of amortization, as in Owadally and Haberman (1999). In this way, the contributions are proportional to the unfunded liabilities, requiring more amortization effort when the plan is underfunded. The pension plan is stochastic, supposing that benefits follow a geometric Brownian motion as in Josa-Fombellida and Rincón-Zapatero (2004). It is then shown that both the stochastic actuarial liability and the normal cost are also geometric Brownian motions, and a relationship between these variables is found. The riskless rate of interest is supposed to be given by a mean-reverting process, as in Vaşicek (1977). An interesting question addressed in the paper is the selection, according to a valuation criterion, of the technical rate of actualization to value the liabilities. The financial market also comprises a family of zero-coupon bonds of fixed maturity and a risky stock, which are correlated with the source of uncertainty of the benefits.

The results obtained are based on the analytical solutions found by means of the dynamic programming approach. The optimal investment in the bond has four summands: (i) the classical optimal one in Merton (1971); (ii) a positive term decreasing to zero with the terminal date of the plan involving parameters of the riskless rate of interest; (iii) the market price of risk multiplied by an expression involving diffusion coefficients of the bond and the stock and the excess expected return of the stock; and (iv) a term proportional to the actuarial liability that vanishes if there is no correlation in the financial instruments or if the benefits are deterministic. The optimal investment in the risky asset follows a similar pattern, but in this case there is no corresponding term to those described in (i) and (ii).

The paper is organized as follows: Section 2 defines the elements of the pension scheme of an employment system. We suppose the technical rate of interest is random. The actuarial functions are also introduced and we prove a relation between these functions when the benefits are given by a geometric Brownian motion. In Section 3, we explain the financial market structure. In Section 4, we find a risk-neutral valuation of the liabilities, giving rise to an expression for the technical rate of actualization, that relates it with the interest rate and the correlation parameters between the sources of uncertainty, as well as with the parameters defining the stochastic evolution of liabilities. In Section 5, we consider that the fund is invested in a riskless asset (savings account) and in two risky assets (a bond and a stock). We state the problem of minimizing the expected value of the terminal solvency risk and we explicitly solve it. In Section 6, the results are illustrated with a numerical analysis of the problem, analyzing the investment time evolution pattern in the bond and in the stock. Finally, Section 7 is dedicated to establishing some conclusions and possible extensions. All proofs are in Appendix A.

2. The pension model

The pension plan we take into account is an aggregated pension fund of the DB type, thus the benefits are established in advance by the manager. With the objective of the delivery of retirement benefits to the workers, the plan sponsor continuously withdraws time-varying funds. The variables listed below refer to the total group of participants. The principal elements intervening in the funding process and the essential hypotheses allowing its temporary evolution to be determined are as follows.

Notation of the elements of the pension plan

T	planning horizon or date of the end of the pension plan, with $0 < T < \infty$
$F(t)$	value of fund assets at time t
$P(t)$	benefits promised to the participants at time t, which are related to the salary at the moment of retirement
$C(t)$	contribution rate made by the sponsor at time t to the funding process
$AL(t)$	actuarial liability at time t, that is, total liabilities of the sponsor
$NC(t)$	normal cost at time t; if the fund assets match the actuarial liability, and if there are no uncertain elements in the plan, the normal cost is the value of the contributions allowing equality between asset funds and obligations
$UAL(t)$	unfunded actuarial liability at time t, equal to $AL(t) - F(t)$
$M(x) \times 100\%$	percentage of the actuarial value of the future benefits accumulated until age $x \in [a, d]$, where a is the common age of entrance in the fund and d is the common age of retirement for all participants. Function M is a differentiable distribution function on $[a, d]$. In particular, $M(a) = 0$ and $M(d) = 1$
$\delta(t)$	technical rate of actualization. It is the rate of valuation of the liabilities, which can be specified by the regulatory authorities
$r(t)$	risk-free market interest rate.

Josa-Fombellida and Rincón-Zapatero (2004) considers that there exist disturbances affecting the evolution of the benefits P and hence, the evolution of the normal cost NC and the actuarial liability AL, but the rate of valuation δ of the plan is constant. In this paper we add a more general assumption: we suppose the short rate of interest r is random. This means that δ is also random. In order to simplify, we will suppose both processes have the same source of uncertainty. As we have commented in Section 1, three sources of randomness appear in the problem: benefits, interest rate and stock. Thus, to model this situation, we consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where $\mathscr{F} = \{\mathscr{F}_t\}_{t \geqslant 0}$ is a complete and right continuous filtration generated by the three-dimensional standard Brownian motion, $\mathscr{F}_t = \sigma\{(w(u), w_B(u), w_S(u)) : 0 \leqslant u \leqslant t\}$ and $\mathbb{P}$ is a probability measure on Ω. We assume that r and δ satisfy stochastic differential equations depending on w_B only. The benefits randomness is due to another Brownian motion w_P. Given that the benefits P are conditioned by the increase in salary of the sponsoring employees, we suppose the existence of correlation $q_1 \in [-1, 1]$ between the Brownian motions w_P and w_B, and $q_2 \in [-1, 1]$ between Brownian motions w_P and w_S, which can be explained by the effects of salary on inflation and the effects of the latter on the asset prices. This means that $w_P(t) = \sqrt{1 - q_1^2 - q_2^2}\, w(t) + q_1 w_B(t) + q_2 w_S(t)$, for all $0 \leqslant t \leqslant T$, supposing $q_1^2 + q_2^2 \leqslant 1$. When $q_1^2 + q_2^2 < 1$, the risk in the benefits outgo cannot be eliminated by trading in the financial market.

We consider that there r, δ and P are diffusion processes given by the stochastic differential equations

$$
\begin{aligned}
dr(t) &= \mu_r(t, r(t))dt + \eta_r(t, r(t))dw_B(t), \\
d\delta(t) &= \mu_\delta(t, \delta(t))dt + \eta_\delta(t, \delta(t))dw_B(t), \\
dP(t) &= \mu_P(t, P(t))dt + \eta_P(t, P(t))dw_P(t),
\end{aligned}
$$