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Valuing continuous-installment options

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ABSTRACT

Installment options are path-dependent contingent claims in which the premium is paid discretely or continuously in installments, instead of paying a lump sum at the time of purchase. This paper deals with valuing European continuous-installment options written on dividend-paying assets in the standard Black–Scholes–Merton framework. The valuation of installment options can be formulated as a free boundary problem, due to the flexibility of continuing or stopping to pay installments. On the basis of a PDE for the initial premium, we derive an integral representation for the initial premium, being expressed as a difference of the corresponding European vanilla value and the expected present value of installment payments along the optimal stopping boundary. Applying the Laplace transform approach to this PDE, we obtain explicit Laplace transforms of the initial premium as well as its Greeks, which include the transformed stopping boundary in a closed form. Abelian theorems of Laplace transforms enable us to characterize asymptotic behaviors of the stopping boundary close and at infinite time to expiry. We show that numerical inversion of these Laplace transforms works well for computing both the option value and the optimal stopping boundary.

1. Introduction

Installment options or *pay-as-you-go* options are path-dependent claims in which a small amount of up-front premium instead of a lump sum is paid at the time of purchase, and then a sequence of *installments* are paid up to a fixed maturity. The holder has the right of stopping payments at any time, thereby terminating the option contract: If the option is not worth the net present value (NPV) of the remaining payments, she/he does not have to continue to pay further installments. Hence, an optimal stopping problem arises for the installment option even in European style. The option can be exercised only if all installments are paid until maturity. Due to the additional right to terminate payments, the total premium charged for an installment option is greater than that for a standard option. An installment option with payments at pre-specified dates is usually referred to as a *discrete-installment* option, whereas its continuous-time limit in which premium is paid at a certain rate per unit time is referred to as a *continuous-installment* option. This paper deals with a European-style continuous-installment option.

In actual markets, installment options have been traded actively, e.g., installment warrants on Australian stocks listed on the Australian stock exchange (ASX) (Ben-Ameur et al., 2005, 2006), a 10-year warrant with 9 annual payments offered by Deutsche Bank (Davis et al., 2001) and so on. Also, many life insurance contracts and capital investment projects can be thought of as installment options. For example, Majd and Pindyck (1987) developed a model for optimal sequential investment, in which a firm invests continuously until the project is completed, investment can be stopped and later restarted without paying any additional costs. Their model can be considered as a European-style continuous-installment option where the remaining expenditure required to complete the project is used as a state variable instead of time; see also Dixit and Pindyck (1994, Chapter 10). However, there have been relatively few studies on installment options: For European-style discrete-installment options, the case of two installments is the compound option, which is an option written on an option; see Geske (1977, 1979). Davis et al. (2001, 2002) applied the concept of compound options and NPV to obtain no-arbitrage bounds of the initial call premium in the (possibly more general) Black–Scholes–Merton framework (Black and Scholes, 1973; Merton, 1973), and then to examine dynamic and static hedging strategies. They intuitively showed that holding an installment call option is equivalent to holding an associated European call option with the same payoff plus the right to sell this option at any installment date at a price equal to the NPV of all future installments. The latter can be understood as an American compound put option written on the vanilla call option, where all the maturity dates are same. Griebsch et al. (2007) proved this intuitive idea on premium decomposition to be correct; see also Wystup et al. (2004). For American-style discrete-installment options, Ben-Ameur et al.

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(2006) developed a DP algorithm for computing the option value approximated by a piecewise-linear interpolation, which is applied to valuing ASX installment warrants with dilution effects.

For continuous-installment options, Ciurlia and Roko (2005) analyzed the American case approximately by applying the multi-piece exponential function (MEF) method to an integral representation of the initial premium. To check the accuracy of the MEF approximation, they executed numerical comparisons with benchmark results obtained by the finite-difference method as well as a Monte Carlo method. The MEF method has been originated by Ju (1998) in the valuation of the standard American put option, which generates a piecewise continuous, i.e., *discontinuous* optimal stopping and early exercise boundaries. This discontinuity is a serious obstacle to decision-makings of the option holder. As for the European case, Alobaidi et al. (2004) used an integral transform to solve a free boundary problem due to the flexibility of continuing or stopping to pay installments, obtaining asymptotic properties of an optimal stopping boundary close to maturity. However, their method is *not* appropriate for quantitative valuation, because the integral transform adopted there is too special to invert it numerically. The target of this paper is also a European continuous-installment option written on a dividend-paying asset in the setup of the standard Black–Scholes–Merton framework, to which we apply a Laplace transform approach; see Kimura (2007) for the American case.

This paper is organized as follows: In Section 2, on the basis of a partial differential equation (PDE) for the values of the continuous-installment call/put options, we derive an integral representation for each initial premium, being expressed as a difference of the corresponding European vanilla value and the expected present value of installment payments along the optimal stopping boundary. In Section 3, applying the Laplace transform approach to the PDE, we obtain explicit Laplace transforms of the initial premium as well as its Greeks, which include the transformed stopping boundary in a closed form. We prove that the Laplace transform of the initial premium can be decomposed into those of the associated vanilla option and its American compound put option. Abelian theorems of Laplace transforms enable us to characterize asymptotic behaviors of the stopping boundary. In Section 4, we show some computational results for particular cases with the aid of numerical Laplace transform inversion. Finally, in Section 5, we conclude and give further remarks as well as directions of future research.

2. Integral representation

Let $(S_t)_{t\geqslant 0}$ be the price process of the underlying asset. Assume that $(S_t)_{t\geqslant 0}$ is a risk-neutralized diffusion process described by the linear stochastic differential equation (SDE)

$$\frac{\mathrm{d}S_t}{S_t} = (r-\delta)\mathrm{d}t + \sigma \mathrm{d}W_t, \quad t \geqslant 0, \tag{1}$$

where $r > 0$ is the risk-free rate of interest, $\delta \geqslant 0$ is the continuous dividend rate, and $\sigma > 0$ is the volatility coefficient of the asset price. In (1), $W \equiv (W_t)_{t\geqslant 0}$ denotes a one-dimensional standard Brownian motion process on a filtered probability space $(\Omega, (\mathscr{F}_t)_{t\geqslant 0}, \mathscr{F}, \mathbb{P})$ where $(\mathscr{F}_t)_{t\geqslant 0} \equiv \mathbb{F}$ is the natural filtration corresponding to W and the probability measure $\mathbb{P}$ is chosen so that the stock has mean rate of return r. In addition, let $q > 0$ be the continuous-installment rate, which means the holder pays an amount $q\,\mathrm{d}t$ in time $\mathrm{d}t$, while the asset itself pays a continuous dividend in the amount of $\delta S_t\,\mathrm{d}t$ to the holder at the same time.

Let t $(\geqslant 0)$ be the purchase time of a continuous-installment option. Then, the initial premium $V \equiv V(t, S_t; q)$ of this option is a function of the time t, the current asset value $S_t \equiv S$, and the continuous-installment rate q. From the standard argument of constructing the hedged portfolio consisting of one option and an amount $-\frac{\partial V}{\partial S}$ of the underlying asset, we see that the initial premium V satisfies an inhomogeneous PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta)S\frac{\partial V}{\partial S} - rV = q. \tag{2}$$

See Ciurlia and Roko (2005) for details. If $q = 0$, then the homogeneous equation agrees with the so-called Black–Scholes–Merton PDE.

2.1. Call case

Consider a European-style installment call option with maturity date T and strike price K. The payoff at the maturity is given by $(S_T - K)^+$, where $(x)^+ = \max(x, 0)$. Let $c \equiv c(t, S_t; q)$ denote the value of the continuous-installment call option at time $t \in [0, T]$. In the absence of arbitrage opportunities, the value $c(t, S_t; q)$ is a solution of an *optimal stopping problem*

$$c(t,S_t;q) = \operatorname*{ess\,sup}_{\tau_c\in[t,T]} \mathbb{E}\left[\mathbf{1}_{\{\tau_c\geqslant T\}}\mathrm{e}^{-r(T-t)}(S_T-K)^+ - \frac{q}{r}\left(1-\mathrm{e}^{-r(\tau_c\wedge T-t)}\right)\Big|\mathscr{F}_t\right] \tag{3}$$

for $t \in [0, T]$, where $a \wedge b = \min\{a, b\}$, τ_c is a stopping time of the filtration $(\mathscr{F}_t)_{t\geqslant 0}$, and the conditional expectation is calculated under the risk-neutral probability measure $\mathbb{P}$. The random variable $\tau_c^* \in [t, T]$ is called an *optimal stopping time* if it gives the supremum value of the right-hand side of (3). If $q \leqslant 0$, then $\tau_c^* = T$ (a.s.), i.e., it is optimal *not* to stop paying installments before the maturity.

Solving the optimal stopping problem (3) is equivalent to finding the points (t, S_t) for which termination of the contract is optimal. Let $\mathscr{D} = [0, T] \times [0, +\infty)$, and $\mathscr{S}$ and $\mathscr{C}$ denote the *stopping region* and *continuation region*, respectively. In terms of the value function $c(t, S_t; q)$, the stopping region $\mathscr{S}$ is defined by

$$\mathscr{S} = \{(t, S_t) \in \mathscr{D} \,|\, c(t, S_t; q) = 0\},$$

for which the optimal stopping time τ_c^* satisfies

$$\tau_c^* = \inf\{u \in [t, T] \,|\, (u, S_u) \in \mathscr{S}\}.$$

The continuation region $\mathscr{C}$ is the complement of $\mathscr{S}$ in $\mathscr{D}$, i.e.,

$$\mathscr{C} = \{(t, S_t) \in \mathscr{D} \,|\, c(t, S_t; q) > 0\}.$$

The boundary that separates $\mathscr{S}$ from $\mathscr{C}$ is referred to as a *stopping boundary* (or a *cancellation boundary*), which is defined by

$$\underline{S}_t = \inf\{S_t \in [0, +\infty) \,|\, c(t, S_t; q) > 0\}, \quad t \in [0, T]. \tag{4}$$

Since $c(t, S_t; q)$ is nondecreasing in S_t, the stopping boundary $(\underline{S}_t)_{t\in[0,T]}$ is a *lower* critical asset price below which it is advantageous to terminate the option contract by stopping the payments, and it vanishes when $q \leqslant 0$, i.e., $\underline{S}_t \equiv 0$ for $t \in [0, T]$.

In the continuation region $\mathscr{C}$, the call value $c(t, S; q)$ $(S \equiv S_t$ for abbreviation) is obtained by solving the inhomogeneous PDE

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} + (r-\delta)S\frac{\partial c}{\partial S} - rc = q, \quad S > \underline{S}_t, \tag{5}$$

with the boundary conditions

$$\left|\begin{array}{l} \lim\limits_{S\downarrow \underline{S}_t} c(t,S;q) = 0, \\ \lim\limits_{S\downarrow \underline{S}_t} \dfrac{\partial c}{\partial S} = 0, \\ \lim\limits_{S\uparrow\infty} \dfrac{\partial c}{\partial S} < \infty. \end{array}\right. \tag{6}$$

The first (*value matching*) condition implies that the initial premium is continuous across the stopping boundary, and the second (*smooth pasting*) condition further implies that the slope is continuous. The terminal condition is clearly given by

$$c(T, S; q) = (S-K)^+. \tag{7}$$