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Interfaces with Other Disciplines

Kriging of financial term-structures

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ABSTRACT

Due to the lack of reliable market information, building financial term-structures may be associated with a significant degree of uncertainty. In this paper, we propose a new term-structure interpolation method that extends classical spline techniques by additionally allowing for quantification of uncertainty. The proposed method is based on a generalization of kriging models with linear equality constraints (market-fit conditions) and shape-preserving conditions such as monotonicity or positivity (no-arbitrage conditions). We define the most likely curve and show how to build confidence bands. The Gaussian process covariance hyper-parameters under the construction constraints are estimated using cross-validation techniques. Based on observed market quotes at different dates, we demonstrate the efficiency of the method by building curves together with confidence intervals for term-structures of OIS discount rates, of zero-coupon swaps rates and of CDS implied default probabilities. We also show how to construct interest-rate surfaces or default probability surfaces by considering time (quotation dates) as an additional dimension.

1. Introduction

Constructing term-structures is at the heart of asset pricing and risk management. A term-structure is a curve which describes the evolution of some financial or economic quantities as a function of time horizon. Typical examples are the term-structure of risk-free interest-rates, the term-structure of bond yields or credit spreads, the term-structure of default probabilities or the term-structure of stock return implied volatilities. These curves are typically not directly observed in the market. Thus, the curve construction is based on a benchmark set of contingent financial instruments whose values explicitly depend on some part of the curve. In practice, market quotes of these products only provide a partial information on the term-structure since they can only be considered to be reliable for a small set of liquid maturities. The problem is then to transform a small set of market quotes into a continuum set of values representing the evolution of the underlying quantity of interest with respect to time horizon.

On practical grounds, the curve is assumed to belong to a family of parametric functions (Nielson–Siegel functional Nelson and Siegel, 1987, polynomial splines, Smith & Wilson, 2001) and its construction consists in finding the underlying parameters that best fits observed market quotes for all available maturities. In de Andrés Sánchez and Gómez (2004), the interest-rate term-structure is estimated from bid-ask spreads of underlying instruments using fuzzy regression techniques. Hagan and West (2006) provide a review of different interpolation techniques for curve construction. They introduce a monotone convex method and postulate a series of quality criterion such as ability to fit market quotes, arbitrage-freeness, smoothness, locality of interpolation scheme, stability of forward rate and consistency of hedging strategies. Andersen (2007) analyzes the use of hyperbolic tension splines for construction of interest-rate term structures. The underlying optimization allows the user to control the relative importance of fit precision with respect to shape preservation (smoothness of the curve, penalization of oscillations and excess convexity/concavity). In the same vein, Chiu, Fang, Lavery, Lin, and Wang (2008) shows that L_1 cubic splines minimizes the curve oscillation without sacrificing good approximation of the data. Iwashita (2013) makes a survey of non-local spline interpolation techniques which preserve stability of forward rates. Other papers such as Ametrano and Bianchetti (2009), Chibane, Selvaraj, and Sheldon (2009), Kenyon and Stamm (2012) or Fries (2013) are concerned with the adaptation of curve construction methods in a multi-curve interest-rate environment. Other models for the term structure of interest rates are also available in the Operational Research literature (see, e.g., Boero and Torricelli (1996), Mercurio and Moraleda (2000), Schmidt (2011), Moreno and Platania (2015), Renne (2016)). Note that, in terms of interpolation scheme, there is no consensus towards a particular best practice method in all circumstances. In addition, the previous approaches does not account

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for the uncertainty embedded in the process of curve construction. This could be of primary importance given that the market inputs may be unreliable or even inexistent for some maturities.

This issue is related to the study of model uncertainty and its impact on risk management. This topic has been studied since a certain time period and, following the recent financial crisis, has received a particular interest. Impact of model risk on valuation and hedging of financial derivatives have been treated by, among others, Derman (1996), Eberlein and Jacod (1997), El Karoui, Jeanblanc-Picquè, and Shreve (1998), Green and Figlewski (1999), Branger and Schlag (2004), Cont (2006), Davis and Hobson (2007), Henaff (2010), Morini (2011). In most papers, the question of model risk is restricted to the class of derivative products. One of the main objective is to quantify model uncertainty, for instance to obtain bounds for the arbitrage free value of some derivative instruments, given some information on the underlying securities, such as marginal distribution of its price at some particular time horizons. In contrast, the question of model risk embedded in the construction of marginal distributions or term-structure functions themselves has not been investigated as a main object, whatever it may concern discount curves, zero-coupon curves, swap basis curves, bond term structures or CDS-implied survival curves.

From Kimeldorf and Wahba (1970) and Mardia, Kent, Goodall, and Little (1996), it is well-known that spline fitting is a special case of kriging (see also Bay, Grammont, & Maatouk, 2015; Bay, Grammont, & Maatouk, 2016). In addition, kriging allows to account for quantification of uncertainty. Kriging has been developed in geosatistics to estimate the density of some mineral resource in the ground given a relatively small set of borehole, see Krige (1951), Matheron (1963), Cressie (1990). Its principle relies on the determination of the conditional distribution of a spatial random field given a set of observed values of the field. The main interest of this method is that it allows to build a predictor of quantities of interest at other locations, as well as uncertainties relying on this prediction.

Kriging is now widely used in many fields like hydrology, air pollution, epidemiology, weather prediction, etc. to interpolate some quantity of interest given some known values at different locations. Despite its popularity, there are relatively few works concerning kriging in actuarial sciences or in finance. Many reference academic journals of these fields give only few or even no entry corresponding to the word “kriging”. The method is however sometimes referred to using the terms “Gaussian Processes” and “machine learning”. Some existing works using kriging methodology in actuarial sciences and finance concern for example dynamic lifetime adjustments (Debón, Martínez-Ruiz, & Montes, 2010), variable annuities valuation (Gan & Sheldon Lin, 2015; Guojun, 2013), nested simulation of expected shortfall (Liu & Staum, 2010), Vasicek model calibration (Sousa, Esquível, & Gaspar, 2012), stock market linkages (Asgharian, Hess, & Liu, 2013) or credit scoring (Fernandes & Artes, 2016). Other works using spatial techniques are Kanevski, Maignan, Pozdnoukhov, and Timonin (2008) on interest rates, Benth (2015) for energy futures prices. Some preprints or conference papers also mention the fit of some financial models (Stutvoet, 2007), spatial insurance (Paulson & Hart, 2006), trading and hedging strategies (Baysal, Nelson, & Staum, 2008), valuation of Bermudan options (Ludkovski, 2015). Kriging methods naturally rely on some assumptions on the underlying random fields, and one must carefully consider all conditions that must be satisfied before constructing a kriging model.

In practice, the term-structure under construction has to satisfy several type of conditions. One of the most important condition is the compatibility of the curve with market data, i.e., if the curve is used to value a benchmark set of instruments (under a specific pricing rule), the resulting values shall be as close as possible to the observed market quotes. In many classical situations, the market-fit condition translates into a system of linear constraints which can be easily incorporated in kriging techniques. In addition, kriging can also handle the presence of noisy observations (using the so-called *nugget effect*). This may be relevant in situation where, due to the lack of liquidity, market quotes cannot be considered to be reliable. It is then possible to incorporate an additional level of uncertainty (degree of confidence) associated with market observations. Monotonicity constraints also appears to be important in many applications. For instance, the price of default-free zero-coupon bonds (or risk-free discount factors) is a non-increasing function of time-to-maturities under no-arbitrage assumption. Survival functions inferred from CDS spread term-structures are [0, 1]-valued non-increasing functions.

Recently, some authors have studied the integration of monotonicity constraints into Gaussian process emulators, see e.g. Golchi, Bingham, Chipman, and Campbell (2015) and Kleijnen and Van Beers (2013). However, these methods do not guaranty monotonicity constraints in the entire domain. The article Abrahamsen and Benth (2001) also deals with the introduction of constraints at some location of a Gaussian process. In Maatouk and Bay (2014), classical kriging has been improved to tackle monotonicity, positivity constraints or bounds constraints on the curve values. In the present paper, we show how “constrained” kriging techniques can be used to extend the classical spline interpolation approaches by additionally quantifying the uncertainty in some illiquid part of the curve.

The paper is organized as follows. Section 2 states the term-structure construction problem and gives specific examples of market-fit conditions and shape preserving constraints. In Section 3, we briefly recall Gaussian process modeling with interpolation conditions or with more general linear equality constraints. In Section 4, we present the model defined in Maatouk and Bay (2014) to incorporate monotonicity constraints into a Gaussian process emulator. We then study the associated properties such as convergence to the constrained interpolation spline and the estimation of the covariance hyper-parameters. In Section 5, based on real market data, we construct curves together with confidence intervals for different financial term-structures such as OIS discount curves, zero-coupon swap curves and CDS-implied default distributions.

2. The term-structure construction problem

The main ingredients in the construction of a term-structure function is a set of financial products whose value depends on some points of the curve. Then, observing the price of these products provides an indirect (and partial) information on the curve. The first step is then to specify the relation between the value of these products and the values of the curve at different time horizons. In this paper, we restrict ourselves to situations where this relation is linear. As we will see, this is the case in many practical situations such as the construction of corporate or sovereign bond yield curve, the construction of OIS discount curves, the construction of forward curves based on fixed-vs.-floating interest-rate swaps or the construction of implied default rates based on CDS spreads.

2.1. Market-fit and shape-preserving conditions

The aim is to construct at some quotation date t a term-structure function $T \to P(t, T)$, based on the observation of a series of market quotes $S_1(t) \dots, S_n(t)$ corresponding to the market value of n financial instruments with time-to-maturities $T_1, \dots, T_n$. In what follows, the quantity T denotes a time length (as opposed to a calendar date), so that $P(t, T)$ corresponds to the value of the curve at time horizon T or at calendar date $t + T$. The observation