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Discrete Optimization

Exact and heuristic algorithms for the Hamiltonian p-median problem

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ABSTRACT

This paper presents an exact algorithm, a constructive heuristic algorithm, and a metaheuristic for the *Hamiltonian p-Median Problem* (HpMP). The exact algorithm is a branch-and-cut algorithm based on an enhanced *p*-median based formulation, which is proved to dominate an existing *p*-median based formulation. The constructive heuristic is a giant tour heuristic, based on a dynamic programming formulation to optimally split a given sequence of vertices into cycles. The metaheuristic is an iterated local search algorithm using 2-exchange and 1-opt operators. Computational results show that the branch-and-cut algorithm outperforms the existing exact solution methods.

1. Introduction

This paper studies the *Hamiltonian p-Median Problem* (HpMP), defined on a complete undirected graph $G = (V, E)$, where V is the vertex set, and $E = \{(i, j) : i, j \in V, i < j\}$ is the edge set. There is a cost c_{ij} associated with every edge (i, j). The aim is to partition the graph into p subsets of vertices, each connected by a single cycle. The objective is to minimize the total cost of edges belonging to the cycles. Following the convention of Gollowitzer, Gouveia, Laporte, Pereira, and Wojciechowski (2014), we only consider subsets (cycles) of cardinality 3 or more. The *Traveling Salesman Problem* (TSP) is a special case of the HpMP with $p = 1$, and consequently the HpMP is NP-hard. It is worth mentioning that the 2-matching problem, which returns an arbitrary number of cycles is solvable in polynomial time (see, for example Miller & Pekny, 1995).

Laporte, Nobert, and Pelletier (1983) introduced a series of location-routing problems and provided computational results for exact algorithms using cutting planes. One of these problems was to locate no more than p non-intersecting cycles in a graph with minimum cost, which was the precursor of the HpMP. The HpMP was introduced by Branco and Coelho (1990). It has received relatively little attention from researchers, and the existing studies have mostly focused on exact algorithms. Motivated by an application in laser multi-scanners, (Glaab & Pott, 2000) have studied the HpMP and presented a three-index formulation, together with results on the dimension of the associated polytope. Zohrehbandian (2007) has proposed a formulation for the HpMP based on a three-index *Vehicle Routing Problem* formulation, but did not provide any computational results. Gollowitzer, Pereira, and Wojciechowski (2011) have provided three formulations for the HpMP together with valid inequalities and branch-and-cut algorithms. In a later study, (Gollowitzer et al., 2014) have introduced seven different formulations for the HpMP which they have compared in terms of dominance relationships. They have also presented computational results for up to $|V| = 100$. Hupp and Liers (2013) have conducted a polyhedral analysis of an HpMP formulation using only edge variables and proved that a subset of the well-known 2-matching inequalities from the TSP define facets of the HpMP polytope.

There exist very few studies on heuristics for the HpMP and its variants. Glaab (2002) provided fast heuristics and improved lower bounds for a variant of the HpMP that arises in cutting problems. Uster and Kumar (2010) have studied the *Balanced Ring Problem*, which is another variant of the HpMP requiring the number of vertices on each cycle to be almost equal. They have provided a GRASP-based constructive algorithm as well as a local search heuristic. To the best of our knowledge, no metaheuristics have yet been proposed for the HpMP.

The remainder of this paper is organized as follows. In Section 2, we recall a integer linear programming formulation for the HpMP proposed by Gollowitzer et al. (2014); we also introduce an alternative formulation with several reinforcements and we develop a branch-and-cut algorithm based on this formulation. In Section 3.1, we provide a giant tour heuristic based on a Dynamic Programming formulation. In Section 3.2, we provide an

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Iterated Local Search (ILS) algorithm for the HpMP. In Section 4, we present the computational results for our algorithms on benchmark instances. Conclusions follow in Section 5.

2. Enhanced *p*-median based formulation

Gollowitzer et al. (2014) have proposed a p-median based formulation, which they call Model 3. It uses variables for assigning vertices to other vertices. For the sake of completeness, we present their formulation below, which we call HpMP1. The authors denote the ordered vertex pairs of every edge $(i, j) \in E$ as $\gamma(i, j) = \{(i, j), (j, i)\}$, and the edges between a subset of vertices $W \subset V$ and the remaining vertices as $\delta(W)$. Let x_{ij} be equal to 1 if edge (i, j) belongs to the solution, and 0 otherwise. Let y_i be equal to 1 if it is selected as a depot, and 0 otherwise. Finally, let v_{ij} be equal to 1 if vertex i is assigned to depot j, and 0 otherwise. The formulation is then:

(HpMP1)

$$\text{minimize} \quad \sum_{(i,j)\in E} c_{ij}x_{ij} \tag{obj}$$

subject to

$$\sum_{i\in V} y_i = p \tag{pm1}$$

$$\textstyle\sum_{j\in V\setminus\{i\}} v_{ij} + y_i = 1 \quad i \in V \tag{pm2}$$

$$v_{ij} \le y_j \quad i, j \in V : i \neq j \tag{pm3}$$

$$\textstyle\sum_{j\in\delta(i)} x_{ij} = 2 \quad i \in V \tag{deg}$$

$$\textstyle\sum_{(i,j)\in\delta(W)} x_{ij} \ge 2\sum_{l\in V\setminus W} v_{kl} \quad W \subset V, k \in W \tag{pm $\le$}$$

$$v_{ka} + x_{ij} \le 1 + v_{la} \quad (i, j) \in E, (k, l) \in \gamma(i, j), a \in V \setminus \{k, l\} \tag{pm $\ge$}$$

$$y_k + x_{ij} \le 1 + v_{lk} \quad (i, j) \in E, (k, l) \in \gamma(i, j) \tag{pm $\ge\prime$}$$

$$v_{ij} = 0 \quad i, j \in V : i > j \tag{sb}$$

$$x_{ij} \in \{0, 1\} \quad (i, j) \in E \tag{bin}$$

$$v_{ij} \in \{0, 1\} \quad i, j \in V : i \neq j \tag{pm4}$$

$$y_i \in \{0, 1\} \quad i \in V. \tag{pm5}$$

The objective function (obj) minimizes the total cost of cycles. Constraint set (pm1) sets the number of cycles to p. Constraint sets (pm2) and (sb) require every vertex to be assigned to itself or to a vertex having a higher index. Constraint set (pm3) forces a vertex to be chosen as a depot if another vertex is assigned to it. Constraint set (deg) states that every vertex must have a degree of 2 which, in conjunction with (bin), enforces the minimum cycle size to be 3. Constraints (pm $\le$) connect the vertices assigned to the same cycle by forcing two edges between the two complementary subsets if a vertex in one subset is assigned to a vertex in the other subset. Constraints (pm $\ge$) and (pm $\ge\prime$) eliminate connections between vertices that have been assigned to different depots. Constraints (sb) cut off symmetrical solutions by forcing all vertices in a cycle to be assigned to the vertex with the highest index. Finally, (bin), (pm4), and (pm5) are the integrality constraints on the variables.

2.1. Valid inequalities

To facilitate our analysis, we propose an alternative formulation, called HpMP2, for the HpMP. It is obtained by unifying the variables v_{ij} and y_j into the variable w_{ij}, i.e. w_{ij} is a binary variable equal to 1 if and only if vertex i is assigned to vertex j, with $w_{ii} = 1$ meaning that vertex i has been chosen as a depot. This transformation results in a simpler presentation of (pm $\ge$) and (pm $\ge\prime$), and the new sets of inequalities we subsequently propose. For the sake of clarity, we present the resulting formulation in its entirety, including constraints that are not affected by the change of variables:

(HpMP2)

$$\text{minimize} \quad \textstyle\sum_{(i,j)\in E} c_{ij}x_{ij} \tag{1}$$

subject to

$$\sum_{i\in V} w_{ii} = p \tag{2}$$

$$\textstyle\sum_{j\in V} w_{ij} = 1 \quad i \in V \tag{3}$$

$$w_{ij} \le w_{jj} \quad i, j \in V \tag{4}$$

$$\textstyle\sum_{j\in\delta(i)} x_{ij} = 2 \quad i \in V \tag{5}$$

$$\textstyle\sum_{(i,j)\in\delta(W)} x_{ij} \ge 2\sum_{l\in V\setminus W} w_{kl} \quad W \subset V, k \in W \tag{6}$$

$$w_{ka} + x_{ij} \le 1 + w_{la} \quad (i, j) \in E, (k, l) \in \gamma(i, j)\ a \in V \tag{7}$$

$$w_{ij} = 0 \quad i, j \in V : i > j \tag{8}$$

$$x_{ij} \in \{0, 1\} \quad (i, j) \in E \tag{9}$$

$$w_{ij} \in \{0, 1\} \quad i, j \in V. \tag{10}$$

Note that the transformation unifies constraints (pm $\ge$) and (pm $\ge\prime$) into (7). We now state our first result.

Proposition 1. *The following inequalities are valid for HpMP2, and dominate (7):*

$$\sum_{k\in S} w_{ik} + x_{ij} \le 1 + \sum_{k\in S} w_{jk} \quad (i, j) \in E, S \subset V. \tag{11}$$

Proof. Since $\sum_{k\in S} w_{ik} \le 1$ and $x_{ij} \le 1$, this inequality is valid whenever $\sum_{k\in S} w_{ik} = 0$ or $x_{ij} = 0$. Thus, we only have to analyze the case where $\sum_{k\in S} w_{ik} = 1$ and $x_{ij} = 1$. In this case, $x_{ij} = 1$ implies that i and j are assigned to the same depot and hence $\sum_{k\in S} w_{jk} = 1$. Therefore, the inequality (11) is valid. Note that (7) is a special case of (11) if $|S| = 1$ or $|S| = |V| - 1$, and is therefore dominated by (11). □

Although the number of constraints (11) is exponential, these can be separated in polynomial time. For any given edge $(i, j) \in E$, start with $S = \emptyset$ and include a vertex $k \in V$ into S if and only if $w_{ik} > w_{jk}$. This results in an overall complexity of $O(|V|^3)$.

We now focus on (6). Define $\mathcal{F}(W)$ as the set of all sets of pairs (i, j): $i \in W, j \in V\setminus W$ or $i \in V\setminus W, j \in W$ such that for every element of $\mathcal{F}(W)$ there is at most one pair containing any vertex $k \in V$ as its second component. We now state our second result.

Proposition 2. *The following inequalities are valid for HpMP2, and dominate (6):*

$$\sum_{(i,j)\in\delta(W)} x_{ij} \ge 2\sum_{(k,l)\in F} w_{kl} \quad W \subset V, F \in \mathcal{F}(W). \tag{12}$$

Proof. Consider a partition of $\{W, V\setminus W\}$ of V, as depicted in Fig. 1, where the positive x variables are denoted with thin lines, the positive w variables are denoted with arrows, and the partition is denoted by a dashed line. In order to check that constraints (12) are valid, we will prove that a feasible solution of HpMP2, $(\bar{x}, \bar{w})$, satisfies them. Let $F \in \mathcal{F}(W)$. If for a pair $(k, l) \in F$ we have that $\bar{w}_{kl} = 1$, then either vertex $k \in W$ is assigned to depot $l \in V\setminus W$