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Discrete Optimization

Lagrangean relaxation of the hull-reformulation of linear generalized disjunctive programs and its use in disjunctive branch and bound

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ABSTRACT

In this work, we present a Lagrangean relaxation of the hull-reformulation of discrete-continuous optimization problems formulated as linear generalized disjunctive programs (GDP). The proposed Lagrangean relaxation has three important properties. The first property is that it can be applied to any linear GDP. The second property is that the solution to its continuous relaxation always yields 0–1 values for the binary variables of the hull-reformulation. Finally, it is simpler to solve than the continuous relaxation of the hull-reformulation. The proposed Lagrangean relaxation can be used in different GDP solution methods. In this work, we explore its use as primal heuristic to find feasible solutions in a disjunctive branch and bound algorithm. The modified disjunctive branch and bound is tested with several instances with up to 300 variables. The results show that the proposed disjunctive branch and bound performs faster than other versions of the algorithm that do not include this primal heuristic.

1. Introduction

Lagrangean relaxation (or Lagrangean decomposition) of an optimization program is a widely-used and powerful tool to solve problems. The review work by Guignard (2003) discusses how Lagrangean relaxation can be used in different solution methods and applications. Fisher (2004) provides a theoretical background for Lagrangean relaxation of mixed-integer linear programs (MILP). The general idea in the Lagrangean relaxation is to “dualize” some of the constraints in the optimization problem (i.e. remove some constraints from the feasible region of the problem, and penalize the violation of such constraints in the objective function). This approach is particularly useful in problems with complicating constraints. Some of these problems appear in planning (Rong, Lahdelma, & Luh, 2008), scheduling (Terrazas-Moreno, Trotter, & Grossmann, 2011), facility location (Cornuejols, Fisher, & Nemhauser, 1977), and stochastic programming problems (Carøe & Schultz, 1999). In these type of problems, a Lagrangean relaxation is simpler to solve than the original problems.

A particular method that uses Lagrangean relaxation to solve MILPs is the Lagrangean relaxation based branch and bound (Geoffrion, 1974). This method follows the same general idea as the LP based branch and bound, but it solves the Lagrangean relaxation at every node instead of the LP relaxation. This method can be useful in problems in which, by dualizing the complicating constraints, the Lagrangean relaxation is simpler to solve than the LP relaxation. One of the main difficulties in automating this strategy, or any other method that uses Lagrangean relaxation, is identifying the complicating constraints, which can be non-trivial and problem specific. Typically, the modeler needs to identify the problem structure and select the constraints to dualize, or needs to modify the model to allow such a structure (Guignard, 2003).

Linear discrete–continuous optimization problems are typically formulated as MILPs. An alternative framework for modeling these problems is generalized disjunctive programming (GDP) (Raman & Grossmann, 1994). GDP models are used to represent discrete–continuous problems through the use of disjunctions, algebraic equations, and Boolean and continuous variables. Linear GDP problems can be reformulated as mixed-integer linear programs (MILP) and solved with existing MILP solvers. The GDP-to-MILP reformulations are the Big-M (BM) (Wolsey & Nemhauser, 2014), multiple-parameter Big-M (MBM) (Trespalacios & Grossmann, 2015) and Hull reformulation (HR) (Beaumont, 1990; Lee & Grossmann, 2000). The HR reformulations is at least as tight, and typically tighter, than the other two. The downside of the HR is that it yields a larger MILP formulation. Alternatively to the MILP reformulation, GDP problems can be solved with specialized algorithms. In the case of linear GDP problems, the disjunctive branch and bound is a powerful solution method (Beaumont, 1990; Lee & Grossmann, 2000).

In this work we first present a Lagrangean relaxation of the HR for linear GDPs. The proposed Lagrangean relaxation is an MILP,

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and it has three important characteristics. The first one is that the solution to the continuous relaxation of the proposed Lagrangean relaxation always yields 0–1 values for the binary variables of the HR. The second one is that it is easier to solve than the original problem (i.e. the HR). Furthermore, it is easier to solve than the continuous relaxation of the HR. The third one is that this relaxation can be applied to any linear GDP. This means that there is no need to specify which are the complicating constraints in different problems, so automating a method that uses this Lagrangean relaxation can be achieved.

While there are different methods that can make use of the proposed Lagrangean relaxation, in this work we use it to improve the performance of the disjunctive branch and bound algorithm. In particular, we use it as a primal heuristic in the disjunctive branch and bound. A primal heuristic is a method whose purpose is to find good quality feasible solutions quickly (but does not guarantee that such solution is optimal). In the modified disjunctive branch and bound, we evaluate the Lagrangean relaxation at every node and use its solution as primal heuristic for finding feasible solutions to the problem. The continuous relaxation of the Lagrangean relaxation always provides 0–1 values to the binary variables, so the value of the 0–1 variables is fixed and a small LP is solved in search of feasible solutions. The limitation of the proposed Lagrangean relaxation it that the value of its objective function is not better than the value of the objective function in the LP relaxation.

This paper is organized as follows. Section 2 presents a brief background on Lagrangean relaxation of MILPs, generalized disjunctive programming, and the disjunctive branch and bound. Section 3 presents the proposed Lagrangean relaxation of the HR. This section presents the formulation and main properties. The proposed Lagrangean relaxation is then incorporated into a disjunctive branch and bound, which is presented in Section 4. Section 5 demonstrates the performance of the proposed disjunctive branch and bound in an illustrative example. The performance of the disjunctive branch and bound with the Lagrangean relaxation is evaluated against other versions of the disjunctive branch and bound with several instances with up to 300 variables. The results of these instances are presented in Section 6. Finally the conclusions of the paper are presented in Section 7.

2. Background

2.1. Lagrangean relaxation of mixed-integer linear programs

In this section we present a brief review of the Lagrangean relaxation of mixed-integer linear programs. In this work, we consider the complicating constraints to be equality constraints. We refer the reader to the work by Guignard (2003) for a comprehensive review and for proofs of the Theorems and relations presented in this section. Throughout the manuscript, for any given optimization problem (Q) we denote $\nu(Q)$ its optimal value and $F(Q)$ its feasible region.

Without loss of generality, consider the following general mixed-integer linear program:

$$\begin{aligned} & \min c^T x \\ & Ax = b \\ & Cx \le d \\ & x \in X \end{aligned} \tag{P}$$

where X contains the integrality and sign restrictions on x (e.g. $X = \mathbb{R}_+^{n-q} \times \{0,1\}^q$). Consider that $Ax = b$ are the complicating constraints (i.e. the problem becomes much simpler to solve without them). Let λ be a vector of weights, namely the Lagrange multipliers.

The Lagrangean relaxation of (P) is:

$$\begin{aligned} & \min c^T x + \lambda^T (Ax - b) \\ & Cx \le d \\ & x \in X \end{aligned} \tag{$LR1_\lambda$}$$

In ($LR1_\lambda$), the complicating constraints ($Ax = b$) have been "dualized" (i.e. the slacks of the complicating constraints have been added to the objective function, and the complicating constraints dropped from the formulation). Note that if these constraints are inequalities ($Ax \le b$), then the corresponding Lagrange multipliers are non-negative.

It is easy to see that ($LR1_\lambda$) is a relaxation of (P), since $F(P) \subseteq F(LR1_\lambda)$. Therefore, $\nu(LR1_\lambda) \le \nu(P)$ in general.

Theorem 2.1.

1. *If $x(\lambda)$ is an optimal solution of ($LR1_\lambda$) for some λ, ($LR1_\lambda$) is bounded, and the original problem is feasible, then $c^T x(\lambda) + \lambda^T (Ax - b) \le \nu(P)$.*
2. *If in addition $x(\lambda)$ is feasible for (P), then $x(\lambda)$ is an optimal solution of (P), and $c^T x(\lambda) = \nu(P)$*

Theorem 2.1 states that Lagrangean relaxation always provides a lower bound for the MILP problem. The best possible lower bound that the Lagrangean relaxation provides can be obtained with the following optimization problem:

$$\max_{\lambda} \nu(LR1_\lambda) \tag{LD}$$

Problem (LD) is called the Lagrangean dual of (P) with respect to the complicating constraints $Ax = b$.

Let (RP) be the continuous relaxation of (P) defined by omitting the integrality requirements in the set X. In general, $\nu(RP) \le \nu(LD)$. In the particular case in which the Lagrangean dual has the integrality property (i.e. the extreme points of $\{x | Cx \le d\}$ are in X), $\nu(RP) = \nu(LD)$.

In this work, we present a Lagrangean relaxation applicable to the MILP reformulation of problems formulated as GDPs. In the next section, we introduce the GDP formulation and the two main GDP-to-MILP reformulations.

2.2. Linear generalized disjunctive programming

GDP is an alternative framework for modeling discrete–continuous optimization problems. In this section we present the formulation for linear GDPs, as well as the two traditional GDP-to-MILP reformulations: the BM and the HR. For a comprehensive review on formulating GDP problems, as well as the theory for general nonlinear GDPs, we refer the reader to Grossmann and Trespalacios (2013).

The general linear GDP formulation can be represented as follows:

$$\begin{aligned} & \min c^T x \\ \text{s.t.} \quad & Gx \le g \\ & \bigvee_{i \in D_k} \begin{bmatrix} Y_{ki} \\ A^{ki} x \le a^{ki} \end{bmatrix} \quad k \in K \\ & \underset{i \in D_k}{\veebar} Y_{ki} \quad k \in K \\ & \Omega(Y) = True \\ & x^{lo} \le x \le x^{up} \\ & x \in \mathbb{R}^n \\ & Y_{ki} \in \{True, False\} \quad k \in K, i \in D_k \end{aligned} \tag{GDP}$$

In (GDP), the objective is to minimize a linear function of the continuous variables $x \in \mathbb{R}^n$. The global constraints of the problem