Available online at www.sciencedirect.com
ScienceDirect
European Journal of Operational Research 176 (2007) 1334–1347
EUROPEAN JOURNAL OF OPERATIONAL RESEARCH
www.elsevier.com/locate/ejor

Continuous Optimization

A constrained optimization approach to solving certain systems of convex equations

Daniel Solow *, Hantao Li

Department of Operations, Weatherhead School of Management, Case Western Reserve University, Cleveland, OH 44106, United States

Received 15 October 2004; accepted 20 October 2005
Available online 25 January 2006

Abstract

This research presents a new constrained optimization approach for solving systems of nonlinear equations. Particular advantages are realized when all of the equations are convex. For example, a global algorithm for finding the zero of a convex real-valued function of one variable is developed. If the algorithm terminates finitely, then either the algorithm has computed a zero or determined that none exists; if an infinite sequence is generated, either that sequence converges to a zero or again no zero exists. For solving n-dimensional convex equations, the constrained optimization algorithm has the capability of determining that the system of equations has no solution. Global convergence of the algorithm is established under weaker conditions than previously known and, in this case, the algorithm reduces to Newton's method together with a constrained line search at each iteration. It is also shown how this approach has led to a new algorithm for solving the linear complementarity problem.

Keywords: Nonlinear programming; Convex programming; Nonlinear equations; Convex equations

1. Introduction and notation

Due to its importance in solving problems, much effort has been devoted to developing algorithms for finding a zero of an $(n \times n)$ system of nonlinear equations (see [15,2,7]). In this work, a new constrained optimization approach is proposed. The corresponding algorithm can sometimes solve problems that other methods cannot—for instance, several examples are provided where the proposed algorithm succeeds and Newton's method fails. However, the primary advantages of the proposed algorithm are realized when all

* Corresponding author. Fax: +1 216 368 6250.
E-mail address: daniel.solow@case.edu (D. Solow).

0377-2217/$ - see front matter
doi:10.1016/j.ejor.2005.10.041

of the equations are convex. Although less common, there are problems that give rise to the need to solve such a system. One of the most important applications is the linear complementarity problem (see [4]) which, given an $(n \times n)$ matrix M and an n-vector q, is the problem of finding two n-vectors w and x such that:

$$\begin{aligned} &(1) \quad w = Mx + q, \\ &(2) \quad w, x \geqslant 0, \\ &(3) \quad w^T x = 0. \end{aligned} \tag{LCP}$$

Mangasarian [10] showed that this problem is equivalent to finding a zero of the piecewise-linear convex function $f : \mathbb{R}^n \to \mathbb{R}^n$, in which each coordinate function $f_i : \mathbb{R}^n \to \mathbb{R}^1$ is defined by $f_i(x) = \max\{-(Mx + q)_i, -x_i\}$. For this reason, others have studied the problem of solving a system of convex equations. For example, Eaves [6] proposes a homotopic approach for solving piecewise-linear convex equations.

An algorithm is developed here to solve a constrained optimization problem associated with a system of differentiable convex equations. Ortega and Rheinboldt [12] and more extensively Moré [11] propose variants of Newton's method to solve such systems. Although applications of differentiable convex systems appear to be scarce, several applications from the literature are solved in Section 5.1 with the proposed algorithm but, for the most part, the work here is of theoretical interest. Specifically, the advantages of the proposed approach include the ability: (1) to detect that the system has no solution; (2) to establish global convergence under weaker conditions than previously known; and (3) to extend the algorithm from differentiable to piecewise-linear convex equations (and so, to the LCP as a special case).

In Section 2, a global algorithm is developed for finding a zero of a real-valued convex function of one variable or determining that no such point exists. The results in Section 2 are generalized in Section 3 to construct a constrained optimization algorithm for solving certain $(n \times n)$ systems of convex equations. Section 4 deals with computational implementation and conditions for convergence. Algorithmic performance and applications are presented in Section 5, which also includes a comparison of this approach with others for finding a zero of a system of nonlinear equations.

For the most part, standard vector, matrix, and sequence notation is used. With regard to vectors, subscripts refer to components and superscripts are used for sequences. The vector whose coordinates are all 1 is denoted by e. If a sequence of vectors $\{x^k\}$ in $\mathbb{R}^n$ converges to $x \in \mathbb{R}^n$ along a subsequence K, it will be written as $\{x^k\} \to x$ as $k \in K$. It is assumed that the reader is familiar with basic notions associated with a convex function $f : \mathbb{R}^n \to \mathbb{R}^1$ and in particular the gradient and subgradient inequalities [the set of all subgradients of f at x is denoted by $\partial f(x)$]. The reader is referred to Rockafellar [16] for an in-depth discussion of convex functions. Finally, if V is a non-empty subset of $\mathbb{R}^n$, then $(V)^*$ denotes the set of all non-empty subsets of V. Let S and T be non-empty sets. A point-to-set map $M : S \to (T)^*$ is said to be *closed at* $x \in S$ if whenever $\{x^k\} \to x$, $y^k \in M(x^k)$ for all k, and $\{y^k\} \to y$, it follows that $y \in M(x)$. M is *closed* if it is closed at each $x \in S$. With these notations, the algorithms and their proofs of convergence can be presented.

2. A global algorithm for finding a zero of a convex function on R^1

A modified Newton algorithm is proposed for finding a zero of a convex function $h : \mathbb{R}^1 \to \mathbb{R}^1$ or determining that no such point exists. If the algorithm terminates finitely, then either a zero of h is produced or no such point exists. On the other hand, if the algorithm generates an infinite sequence of points, then either the sequence converges to a zero of h or one can conclude that no zero of h exists. The algorithm proceeds like Newton's method, except that the derivative is replaced with an arbitrary subgradient. The steps are as follows: