European Journal of Operational Research 250 (2016) 555–565

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

Cooperation among agents with a proximity relation

J. R. Fernández, I. Gallego, A. Jiménez-Losada*, M. Ordóñez

Escuela Técnica Superior de Ingeniería, Matemática Aplicada II, Universidad de Sevilla, Camino de los Descubrimientos, Sevilla 41092, Spain

ARTICLE INFO

Article history:
Received 5 May 2015
Accepted 17 September 2015
Available online 25 September 2015

Keywords:
Cooperative game
Fuzzy relations
Proximity relations
Choquet integral
Owen value

ABSTRACT

A cooperative game consists of a set of players and a characteristic function determining the maximal gain or minimal cost that every subset of players can achieve when they decide to cooperate, regardless of the actions that the other players take. The relationships of closeness among the players should modify the bargaining among them and therefore their payoffs. The first models that have studied this closeness used a priori unions or undirected graphs. In the a priori union model a partition of the big coalition is supposed. Each element of the partition represents a group of players with the same interests. The groups negotiate among them to form the grand coalition and later, inside each one, players bargain among them. Now we propose to use proximity relations to represent leveled closeness of the interests among the players and extending the a priori unions model.

1. Introduction

Cooperative game theory studies situations where a set of agents (players) bargain for a fair allocation of a common profit resulting from their collaboration, namely a vector with the payoff of each player as coordinates (a payoff vector). In order to establish this allocation a number is known for each subset (coalition) of players representing the profit obtained by them and the mapping that assigns these numbers is named the characteristic function of the game. The Shapley value, Shapley (1953), is one of the point solutions for cooperative games mostly used and studied. It is a function obtaining a payoff vector for each game based in a set of reasonable conditions (axioms) which allow us to decide whether this value is or not the best solution for the problem. Several variations of the Shapley value have been proposed for situations where some additional information about the agents is known. Aumann and Dreze (1974) introduced coalition structures. A coalition structure is a partition of the set of players representing the different coalitions obtained at the end of game. Hence there should be no side payment between these final coalitions. This way has been improved by Myerson (1977) considering communication structures. A communication structure is a graph representing the bilateral cooperation possibilities among the agents. In this case the final coalition structure is the set of connected components in the graph but we can also use the information given by the graph about the internal structure of these coalitions. Owen (1977) proposed a different model from that of Aumann and Dreze based in a different interpretation of the coalition structures. He considered that the coalition structure is a partition of the set of players in a priori unions based in the relations among the agents. But these unions are not considered as a final structure but as a starting point for further negotiations. Thus, as in the original Shapley model, the grand coalition is the final coalition structure. This paper focuses on the Owen variation. So, a coalition of players forms a union if they have the same (or close) interests in the game. Owen obtained a Shapley-type solution (the Owen value) taking into account this information to get a fair allocation of the profit of the grand coalition. Later Casajus (2007) proposed a modification of the Owen model in the Myerson sense. That is, we have an a priori union structure and we know how these unions are formed by means of a connected graph in each group. This graph explains the relation of closeness existing among their players. But closeness is usually a leveled property. For instance, political groups can be organized in a priori ideological unions. Considering equal every ideological closeness between two political parties is actually a simplification of the situation. Aubin (1981), Butnariu (1980) and Mares (2001) introduced fuzzy sets to describe leveled participation of the players in the coalitions (fuzzy coalitions) or fuzziness in the worth of the coalitions given by the characteristic function (fuzzy payoffs). The Choquet integral, Choquet (1953), is a powerful tool from the decision theory which is a way of measuring the expected utility of an uncertain event. Tsurumi, Tanino, and Inuiguchi (2001) used the Choquet integral for fuzzy sets in order to define a Shapley-type solution of a family of games with fuzzy coalitions. Jiménez-Losada, Fernández, Ordóñez, and Grabisch (2010) Jiménez-Losada, Fernández, and Ordóñez (2013); Gallego, Fernández, Jiménez-Losada, and Ordóñez (2014) and Gallardo, Jiménez, Jiménez-Losada, and Lebrón (2014) proposed to use fuzzy

* Corresponding author. Tel.: +34 954486170; fax: +34954486165.
E-mail address: ajlosada@us.es, alfajime@hotmail.com (A. Jiménez-Losada).

http://dx.doi.org/10.1016/j.ejor.2015.09.029
0377-2217/

structures as an additional information for a cooperative game. Particularly, they considered a fuzzy graph to study fuzzy communication structures in the Myerson way. Meng, Zhang, and Cheng (2012) analyzed games on fuzzy coalitions with a priori unions.

Now, we propose a more realistic version of the Owen situation. Following Owen (1977) and Casajus (2007) we consider a fuzzy graph where the fuzzy set of links defines the objective bilateral closeness of interests among the players. This structure is actually a known fuzzy binary relation called proximity relation. This fuzzy relation also establishes unions among the agents but these are not disjoint and each union is represented by a communication structure, thus players are asymmetric in them. While Meng et al. (2012) considered fuzzy coalitions but common a priori unions, we will take usual games but leveled closeness. Preliminaries introduce the needed aspects from cooperative games, a priori unions, communication structures and fuzzy sets to understand the paper. Section 3 analyzes the value introduced by Casajus (2007) in a different way, obtaining an axiomatization comparable to the one of the Owen value. In Section 4 we introduce several ways to reduce a proximity relation which are used later. Section 5 defines an Owen-type value for proximity situations, the prox-Owen value and finally in Section 6 we propose an axiomatization of the new value with reasonable axioms in this context.

2. Preliminaries

2.1. Cooperative TU-games

A *cooperative game with transferable utility*, a game since now, is a pair (N, v) where N is a finite set of elements and $v : 2^N \to \mathbb{R}$ is a mapping satisfying $v(\emptyset) = 0$. The elements of N are named players, the subsets of N are said coalitions and v is the characteristic function of the game. We denote as G the set of games. If $(N, v) \in G$ and $S \subseteq N$ then $(S, v) = (S, v_S) \in G$ is a new game where v_S is the restriction of the characteristic function v to 2^S. An example of a game is the *unanimity game* (N, u_T), with $T \subseteq N$ and $T \neq \emptyset$, defined as $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise. If we fix N, the family $\{u_T\colon T \subseteq N\}$ is a basis of the games over N, that is for every (N, v) there are coefficients c_T such that

$$v = \sum_{\{T\subseteq N: T\neq\emptyset\}} c_T u_T. \tag{1}$$

An *allocation rule* is a function ψ over G which determines for each (N, v) a vector $\psi(N, v) \in \mathbb{R}^N$ interpreted as a payoff vector. The *Shapley value* is an allocation rule defined for every $(N, v) \in G$ and $i \in N$ as

$$\phi_i(N, v) = \sum_{\{S\subset N: i\notin S\}} \frac{(|N| - |S| - 1)!\,|S|!}{|N|!}[v(S \cup \{i\}) - v(S)]. \tag{2}$$

This allocation rule satisfies *efficiency*, that is $\sum_{i\in N} \phi_i(N, v) = v(N)$. The Shapley value is also *linear* namely if $(N, v_1), (N, v_2) \in G$ and $a, b \in \mathbb{R}$ then $\phi(N, av_1 + bv_2) = a\phi(N, v_1) + b\phi(N, v_2)$. A *null player* $i \in N$ for a game (N, v) satisfies $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N\backslash\{i\}$. The Shapley value satisfies the *null player axiom* i.e. if i is a null player for (N, v) then $\phi_i(N, v) = 0$. It is said that $i, j \in N$ are *substitutable players* in a game (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N\backslash\{i, j\}$. The *equal treatment axiom* says that if $i, j \in N$ are substitutable players in (N, v) then $\phi_i(N, v) = \phi_j(N, v)$. It is known that the Shapley value is the only allocation rule over G satisfying efficiency, linearity, null player and equal treatment. Moreover these axioms are not redundant.

2.2. A priori unions

A *game with a priori unions* is a triple (N, v, P) where $(N, v) \in G$ and $P = \{N_1, \ldots, N_m\}$ is a partition of N. Players in N_k for each k have similar interests in the game and they use the union in the bargaining to get a fair payoff. The set of games with a priori unions is denoted as GU.

The *Owen value* ω is an allocation rule over GU. It is supposed that players are interested in the grand coalition N but considering the a priori unions as bargaining elements. Let $(N, v, P) \in GU$ with $P = \{N_1, \ldots, N_m\}$. The *quotient game* (M, v^P) with set of players $M = \{1, \ldots, m\}$ is defined as

$$v^P(Q) = v\left(\bigcup_{q\in Q} N_q\right), \forall Q \subseteq M. \tag{3}$$

Let $k \in M$. For each $S \subset N_k$ the partition P_S of $N\backslash(N_k\backslash S)$ is to replace N_k with S. We define (N_k, v_k) as $v_k(S) = \phi_k\big(M, v^{P_S}\big), \forall S \subseteq N_k$. Finally we solve the game in every group using also the Shapley value. So, for each $i \in N$ if $k(i)$ is such that $i \in N_{k(i)}$ then the Owen value is

$$\omega_i(N, v, P) = \phi_i(N_{k(i)}, v_{k(i)}). \tag{4}$$

The Owen value satisfies efficiency, linearity and null player (the same definitions in GU than in G). It also satisfies *equal treatment in a union*, namely if $i, j \in N_k$ for $k \in M$ are substitutable in (N, v) then $\omega_i(N, v, P) = \omega_j(N, v, P)$. Moreover ω satisfies a similar condition with the unions, the *coalitional symmetry*: if $k_1, k_2 \in M$ satisfy that $v(N_{k_1} \cup \bigcup_{q\in Q} N_q) = v(N_{k_2} \cup \bigcup_{q\in Q} N_q)$ for every $Q \subseteq M\backslash\{k_1, k_2\}$ then

$$\sum_{i\in N_{k_1}} \omega_i(N, v, P) = \sum_{j\in N_{k_2}} \omega_j(N, v, P).$$

Owen (1977) showed that ω is the only allocation rule over GU satisfying efficiency, linearity, null player, equal treatment in each union and coalitional symmetry.[1]

2.3. Communication structures

Let $LN = \{\{i, j\} : i, j \in N \text{ and } i \neq j\}$ be the set of unordered pairs of elements in a finite set N. We will use $ij = \{i, j\}$ from now on. A *communication structure* for N is a graph (N, L) where the set of vertices is N and the set of edges $L \subseteq LN$ is the set of feasible communications among them. Hence we identify a communication structure for N with the set of links L. A *game with communication structure* is a triple (N, v, L) where $(N, v) \in G$ and L is a communication structure for N. A cooperative game (N, v) can be identified with the game with communication structure (N, v, LN). The family of games with communication structure is denoted as GC. Let $(N, v, L) \in GC$ be a game with communication structure. A coalition $S \subseteq N$ whose vertices are connected by the links in L is called *connected*. The maximal connected coalitions correspond to the sets of vertices of the connected components of the graph (N, L) and they are denoted as N/L. This family N/L is actually a partition of N. If $S \subseteq N$ is a coalition then $L_S = \{ij \in L : i, j \in S\}$ and $(S, v, L_S) \in GC$ represents the restriction to S of the game and the communication structure. We use $S/L = S/L_S$. Following Myerson (1977), in a communication structure the final coalition structure is formed by the connected components of the graph, and they cannot get beneficial collaborations among them.

The *Myerson value* is an allocation rule for games with communication structure. Given $(N, v, L) \in GC$ Myerson defines a new game $(N, v/L) \in G$ incorporating the information of the communication structure,

$$v/L(S) = \sum_{T\in S/L} v(T) \quad \forall S \subseteq N. \tag{5}$$

The Myerson value is defined as

$$\mu(N, v, L) = \phi(N, v/L). \tag{6}$$

[1] Owen (1977) used symmetry in each union instead of equal treatment in each union, but both axioms are equivalent in a context with efficiency, linearity and null player.