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Invited Review

Monge properties, discrete convexity and applications ☆

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Dedicated to Professor András Prékopa on the occasion of his 75th birthday

Abstract

Starting from Monge's mass transportation problem we review the role Monge properties play in optimization. In particular we discuss transportation problems whose cost functions fulfill a Monge property, Monge sequences, algebraic Monge properties, the recognition of permuted Monge arrays and multidimensional Monge arrays and the connections between Monge properties and discrete convexity. Finally we discuss Prékopa's recent approach using Monge arrays in bounding multivariate probability distribution functions.

Keywords: Monge property; Transportation problem; Greedy algorithm; Discrete convexity; Bounding probability distribution functions

1. Historical remarks

In the year 1781 Monge published a study with the title *Mémoire sur la théorie des déblais et des remblais* in which he considers the transport of soil from one place to another at minimum cost. He splits the mass to be transported in many small particles and associates as cost to each particle its weight multiplied by the length of the transportation path. The total cost is the sum over all particle costs. Monge observed that in an optimal solution the paths on which the transport takes place should not cross due to the quadrilateral inequality. Monge's model can be viewed as a special continuous transportation problem which was later formalized and investigated by Kantorovich [20,21]. This so-called Monge-Kantorovich mass transfer

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problem (MKP) arises in many areas of mathematics, in particular in probability theory and statistics, see Rachev and Rüschendorf [36].

In 1961 Hoffman [19] considered a discrete version of Monge's transportation problem. He showed that the classical Hitchcock transportation problem of linear programming can be solved in a straightforward way, if the cost coefficients fulfill the property

$$c_{ij} + c_{rs} \leqslant c_{is} + c_{rj} \quad \text{for } 1 \leqslant i < r \leqslant m,\ \ 1 \leqslant j < s \leqslant n, \tag{1}$$

which is in a certain sense an analogue of Monge' original observation that transportation paths should not cross. For this reason Hoffman coined the term *Monge property*. A matrix C whose entries fulfill (1) is nowadays called a *Monge matrix*.

Actually, Hoffman [19] considered a slightly more general case, namely that the cost coefficients of the transportation problem form a so-called Monge sequence. We shall address this feature later in Section 4.

Many years later it turned out that the Monge property plays an important role in discrete optimization and computer science. For example, if the distance matrix of a travelling salesman problem fulfills the Monge property, then the problem can be solved in linear time [29]. Moreover, searching and selecting becomes particularly fast in Monge matrices. Aggarwal et al. [1] proposed a very efficient algorithm for computing all row minima of a Monge matrix. All row minima of an $m \times n$ Monge matrix can be found in $O(n)$ time for $m \leqslant n$ and in $O(n(1 + \log(m/n)))$ time for $m > n$. Thus the minimum entry and the maximum entry of an $m \times n$ Monge matrix can be found in $O(n + m)$ time.

Recently, Prékopa and Hou [34] showed that Monge matrices can be used for bounding multivariate probability distribution functions and in that way opened a new area of applications for Monge matrices.

In the next section we discuss Monge matrices and their basic properties. Then we outline the solution of a transportation problem whose cost matrix fulfills a Monge property by the northwest corner rule. In the following sections we deal with generalizations like Monge sequences, algebraic Monge properties, the recognition of permuted Monge arrays and multidimensional Monge arrays. Section 8 connects Monge properties with discrete convexity. Finally we discuss in Section 9 Prékopa's approach for discrete moment problems and the role of Monge arrays in bounding multivariate probability distribution functions.

2. Monge properties

A real $(m \times n)$ matrix $C = (c_{ij})$ is called a Monge matrix, if

$$c_{ij} + c_{rs} \leqslant c_{is} + c_{rj} \quad \text{for } 1 \leqslant i < r \leqslant m,\ \ 1 \leqslant j < s \leqslant n. \tag{1}$$

It is easy to see that for Monge matrices it suffices to require that the Monge property holds for adjacent rows and adjacent columns. In other words, (1) holds if and only if

$$c_{ij} + c_{i+1,j+1} \leqslant c_{i,j+1} + c_{i+1,j} \quad \text{for all } 1 \leqslant i < m,\ \ 1 \leqslant j < n. \tag{2}$$

An immediate consequence of this observation is that it can be tested in $O(nm)$ time whether a given $m \times n$ matrix is a Monge matrix.

Monge matrices arise in many natural situations. For example, $C = (c_{ij})$ is a Monge matrix, if

- $c_{ij} = u_i + v_j$ for arbitrary real numbers u_i, $i = 1, 2, \ldots, m$, and v_j, $j = 1, 2, \ldots, n$.
- $c_{ij} = u_i \cdot v_j$ for increasing real numbers u_i, $i = 1, 2, \ldots, m$, and decreasing real numbers v_j, $j = 1, 2, \ldots, n$.
- $c_{ij} = \min(u_i, v_j)$ for increasing real numbers u_i, $i = 1, 2, \ldots, m$, and decreasing real numbers v_j, $j = 1, 2, \ldots, n$.
- $c_{ij} := d(P_i, Q_j)$ where $P_1, P_2, \ldots, P_m$ and $Q_1, Q_2, \ldots, Q_n$ are disjoint paths on a convex polygon, see Fig. 1. (This is the discrete case of Monge's original observation.)