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Continuous Optimization

Feasibility problems with complementarity constraints[☆]

R. Andreani[a], J. J. Júdice[b,1], J. M. Martínez[a], T. Martini[a,*]

[a] Department of Applied Mathematics, IMECC-UNICAMP, State University of Campinas, Rua Sérgio Buarque de Holanda, 651 Cidade Universitária "Zeferino Vaz", Distrito Barão Geraldo, 13083–859 Campinas SP, Brazil
[b] Instituto de Telecomunicações, University of Coimbra, Coimbra, Portugal

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ABSTRACT

A Projected-Gradient Underdetermined Newton-like algorithm will be introduced for finding a solution of a Horizontal Nonlinear Complementarity Problem (HNCP) corresponding to a feasible solution of a Mathematical Programming Problem with Complementarity Constraints (MPCC). The algorithm employs a combination of Interior-Point Newton-like and Projected-Gradient directions with a line-search procedure that guarantees global convergence to a solution of HNCP or, at least, a stationary point of the natural merit function associated to this problem. Fast local convergence will be established under reasonable assumptions. The new algorithm can be applied to the computation of a feasible solution of MPCC with a target objective function value. Computational experience on test problems from well-known sources will illustrate the efficiency of the algorithm to find feasible solutions of MPCC in practice.

1. Introduction

A Mathematical Programming Problem with Complementarity Constraints (MPCC) (Luo, Pang, & Ralph, 1996; Outrata, Kocvara, & Zowe, 1998; Ralph, 2007) can be defined in the form

$$\text{Minimize } \varphi(x, y, w) \text{ subject to } H(x, y, w) = 0 \text{ and } \min\{x, w\} = 0, \quad (1)$$

where $x, w \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, while $\varphi : \mathbb{R}^{2n+m} \to \mathbb{R}$, and $H : \mathbb{R}^{2n+m} \to \mathbb{R}^r$ are continuously differentiable functions. The feasible set of MPCC will be denoted by D and $\min\{x, w\}$ denotes a vector of components $\min\{x_i, w_i\}$, $i = 1, \ldots, n$. For all $i = 1, \ldots, n$, the variables x_i, w_i are said to be complementary and satisfy:

$$x_i \geqslant 0, \quad w_i \geqslant 0, \quad x_i w_i = 0, \quad i = 1, \ldots, n. \quad (2)$$

MPCC has appeared frequently in optimization models and has significant applications in different areas of science, engineering and economics (Luo et al., 1996; Outrata et al., 1998; Ralph, 2007). Many theoretical and application papers in Operations Research, as well as survey papers on related topics (Bomze, 2012; Chen, 2000; Júdice, 2014; Kovacevic & Pflug, 2014; Lin & Fukushima, 2010), have been devoted to this problem in recent years. For example, transport network models were considered in García-Rodenas and Verastegui-Rayo (2008), Walpen, Mancinelli and Lotito (2015), Wu, Yin and Lawphongpanich (2011), bilevel optimization in Kovacevic and Pflug (2014), variational inequality formulations in Toyasaki, Daniele and Wakolbinger (2014), multiobjective problems with complementarity constraints in Lin, Zhang and Liang (2013), Ye (2011), electricity markets in Ehrenmann and Neuhoff (2009), Guo, Lin, Zhang and Zhu (2015), Hu and Ralph (2007), Yao, Oren and Adler (2007), quadratic programming with complementarity constraints in Ralph and Stein (2011), optimality conditions in Pang (2007), order-value applications in Andreani, Dunder and Martínez (2005), and oligopolistic equilibrium in Yao, Adler and Oren (2008), among others.

Clearly, MPCC can be seen as a Nonlinear Programming Problem where the n complementarity constraints $\min\{x_i, w_i\} = 0$ are replaced with (2) or even with $x^\top w = 0, x \geqslant 0, w \geqslant 0$. Attempts for solving MPCC by means of nonlinear programming algorithms present some difficulties, mainly because these algorithms may converge to points from which there exist obvious first-order descent directions. This issue is a consequence of the so-called double zeros or biactive indices, i.e., feasible points satisfying at least a constraint $x_i w_i = 0$

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* Corresponding author. Tel.: +551935215988.
E-mail addresses: andreani@ime.unicamp.br (R. Andreani), Joaquim.Judice@co.it.pt (J.J. Júdice), martinez@ime.unicamp.br (J.M. Martínez), tiaramartini@gmail.com (T. Martini).

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with both variables x_i and w_i equal to zero. These difficulties have motivated much research on weak forms of stationarity (Ferris & Pang, 1997; Hoheisel, Kanzow, & Schwartz, 2013; Luo et al., 1996; Outrata et al., 1998; Ralph, 2007; Scheel & Scholtes, 2000) and several algorithms have been designed to compute such weak stationary points (Anitescu, 2005; Anitescu, Tseng, & Wright, 2007; Benson, Sen, Shanno, & Vanderbei, 2006; Fang, Leyffer, & Munson, 2012; Fletcher & Leyffer, 2004; Fukushima, Luo, & Pang, 1998; Fukushima & Tseng, 2002; Hoheisel et al., 2013; Hu & Ralph, 2004; Jiang & Ralph, 2003; Júdice, Sherali, Ribeiro, & Faustino, 2007, Leyffer, López-Calva, & Nocedal, 2006; Luo et al., 1996; Outrata et al., 1998; Ralph, 2007).

In this paper, we will discuss how to compute a feasible solution of the MPCC, that is, a solution of the following Horizontal (possibly nonlinear) Complementarity Problem (HNCP) Gowda (1995):

$$\begin{bmatrix} H(x,y,w) \\ x_1 w_1 \\ \vdots \\ x_n w_n \end{bmatrix} = 0, \; x \geq 0, \; w \geq 0. \tag{3}$$

We will assume that $r \leq m + n$, so that the number of equations in (3) is smaller than or equal to the number of unknowns. The case in which $r = m + n$ has been studied in Andreani, Júdice, Martínez and Patrício (2011b). The case of H affine has been thoroughly discussed in the literature (see for instance Júdice (2014) for a recent survey). The HNCP is NP-hard in this case Murty (1988) but there are many MPCCs where finding a single feasible solution can be considered as an easy task Júdice (2014).

The problem of finding a feasible point of MPCC at which the objective function achieves a target value c_t is naturally formulated as follows:

$$\varphi(x,y,w) \leqslant c_t, \;\; H(x,y,w) = 0, \;\; x \geqslant 0, \;\; w \geqslant 0 \;\; \text{and} \;\; x^\top w = 0. \tag{4}$$

Note that the problem (4) can be written as a standard HNCP adding two auxiliary variables v_1 and v_2, as follows:

$$\varphi(x,y,w) + v_1 = c_t, \;\; H(x,y,w) = 0, \;\; v_1 v_2 = 0, \;\; x_i w_i = 0, \\ i = 1,\ldots,n, \;\; v_1 \geq 0, \;\; v_2 \geq 0, \;\; x \geq 0, \;\; \text{and} \;\; w \geq 0. \tag{5}$$

In this paper we will extend the algorithm introduced in Andreani et al. (2011b), which deals with the case $r = n + m$, for the underdetermined HNCP (3) where r may be smaller than $n + m$. The Projected-Gradient Underdetermined Newton-like algorithm (PGUN) combines fast interior-point iterations with projected-gradient steps. A line-search procedure is employed guaranteeing sufficiently reduction of the natural merit function Andreani, Júdice, Martínez and Patrício (2011a) associated to HNCP. This will allow us to establish global convergence of the PGUN algorithm to a solution of HNCP or to a stationary point of the merit function with a positive function value. In this case the algorithm terminates unsuccessfully. Fast local convergence will be established under reasonable hypotheses.

Computational experience with PGUN for solving the HNCP associated to feasible solutions of some MPCC test problems from a well-known collection Leyffer (2000) will show that, for many instances, projected-gradient iterations are seldom used and the algorithm is able to converge very fast to a solution of HNCP. For other instances, PGUN converges slowly using projected-gradient iterations to a stationary point of the merit function that seems not to be a solution of the HNCP. A practical criterion will be introduced to stop prematurely PGUN and avoid many projected-gradient iterations. As the natural merit function is nonconvex, the choice of the starting point is very important for the success of PGUN. Here we will suggest to restart the PGUN algorithm with a new initial point whenever the criterion mentioned before forced the algorithm to stop prematurely. Numerical results with an implementation of PGUN incorporating these two practical procedures (premature stopping criterion and restarting) show that the method is in general efficient to solve the HNCP and seems to perform better than a Projected Levenberg-Marquardt algorithm Kanzow, Yamashita and Fukushima (2005). We have also tested PGUN for solving (5) associated to a target c_t equal to the best known objective function value of some MPCCs from the collection mentioned before. As discussed in Fernandes, Friedlander, Guedes and Júdice (2001), the introduction of the target constraint to HNCP makes this problem more difficult to tackle and PGUN has more difficulties to solve the HNCP in this case. Despite this, PGUN has been able to provide a target feasible solution of MPCC for the large majority of tested instances.

The organization of this paper is as follows. The properties of the merit function for the HNCP are studied in Section 2. The algorithm PGUN will be described and its global convergence will be analyzed in Section 3. Section 4 will be devoted to the local convergence of the PGUN algorithm. Computational experience with the PGUN algorithm will be reported in Section 5 and some conclusions will be presented in the last section of the paper.

Notation: The 2-norm of vectors and matrices will be denoted by $\|\cdot\|$. If there is no risk of confusion we denote $(x,y,w) = (x^\top, y^\top, w^\top)^\top$, as it has been already done in Section 1. We adopt the usual convention of denoting X the diagonal matrix whose entries are the elements of $x \in \mathbb{R}^n$. The Moore–Penrose pseudoinverse of the matrix A will be denoted by $A\dagger$. The Jacobian matrix of $\Phi\colon \mathbb{R}^n \to \mathbb{R}^m$, with components $\varphi_1, \ldots, \varphi_m$, will be defined by

$$\Phi'(z) = \begin{bmatrix} \frac{\partial \varphi_1}{\partial z_1}(z) & \ldots & \frac{\partial \varphi_1}{\partial z_n}(z) \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi_m}{\partial z_1}(z) & \ldots & \frac{\partial \varphi_m}{\partial z_n}(z) \end{bmatrix}.$$

We define $e = (1, \ldots, 1)^\top$ and

$$\Omega = \{(x,y,w) : x \geqslant 0, w \geqslant 0\}. \tag{6}$$

The Interior of this set will be denoted by $Int(\Omega)$.

2. Stationary points of the sum of squares

The HNCP (3) may be expressed in the form

$$F(x,y,w) = 0, \quad x \geq 0, \quad w \geq 0, \tag{7}$$

where $F : \mathbb{R}^{n+m+n} \longrightarrow \mathbb{R}^{r+n}$ is given by

$$F(x,y,w) = \begin{bmatrix} H(x,y,w) \\ x_1 w_1 \\ \vdots \\ x_n w_n \end{bmatrix}, \tag{8}$$

and $H : \mathbb{R}^{n+m+n} \to \mathbb{R}^r$ has continuous first derivatives.

We define the natural merit function:

$$f(x,y,w) = \|F(x,y,w)\|^2 \tag{9}$$

and we consider the problem

$$\text{Minimize } f(x,y,w) \;\; \text{subject to} \;\; (x,y,w) \in \Omega, \tag{10}$$

where Ω is defined in (6). From now on we will denote $z = (x,y,w)$.

It is well known that, if z^* is an unconstrained stationary point of "Minimize $\|\Phi(z)\|^2$" and the residual $\Phi(z^*)$ is not null, then the rows of the $\Phi'(z^*)$ are linearly dependent. In general, this property is not true in the presence of bound constraints. In what follows, generalizing a result proved in Andreani et al. (2011a), we prove that the non-full-rank property also holds in the case of problem (10) with the definitions (8) and (9).

Theorem 2.1. *Suppose that $\bar{z} = (\bar{x}, \bar{y}, \bar{w}) \in \Omega$ is a stationary point of (10). Then,*