European Journal of Operational Research 249 (2016) 270–280

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Innovative Applications of O.R.

A new elementary geometric approach to option pricing bounds in discrete time models[☆]

Yann Braouezec[a,*], Cyril Grunspan[b]

[a] IESEG School of Management, CNRS-LEM, UMR 9221 Paris campus, Socle de la Grance Arche, 1 Parvis de la Défense, 92044 Paris La Défense Cedex, France
[b] Léonard de Vinci Pôle Universitaire, Finance Lab, 92916 Paris-La Défense cedex, France

ARTICLE INFO

Article history:
Received 10 January 2015
Accepted 17 August 2015
Available online 22 August 2015

Keywords:
Finance
Incomplete markets
Option pricing bounds
Convex hulls
Barycentric coordinates

ABSTRACT

The aim of this paper is to provide a new straightforward *measure-free* methodology based on convex hulls to determine the no-arbitrage pricing bounds of an option (European or American). The pedagogical interest of our methodology is also briefly discussed. The central result, which is elementary, is presented for a one period model and is subsequently used for multiperiod models. It shows that a certain point, called the forward point, must lie inside a convex polygon. Multiperiod models are then considered and the pricing bounds of a put option (European and American) are explicitly computed. We then show that the barycentric coordinates of the forward point can be interpreted as a martingale pricing measure. An application is provided for the trinomial model where the pricing measure has a simple geometric interpretation in terms of areas of triangles. Finally, we consider the case of entropic barycentric coordinates in a multi asset framework.

1. Introduction

What are the no-arbitrage bounds of a European or American put option when a two-period trinomial model is used? What can be said about the evolution of the option pricing bounds as a function of the parameters of the model? Is there any difference between the trinomial and a more general multinomial model? What are the option pricing bounds of a put option if the set of prices is a compact subset of the real line? What are the no-arbitrage bounds of a European digital option when a general continuous-time model is used?

It is the aim of the present paper to develop a straightforward, pricing measure-free approach to answer these questions at the level of difficulty comparable to the popular textbook of Hull (2011). The reason why our approach is straightforward comes from the fact that we derive the pricing bounds of a *given* option. In particular, we don't make any use of the general theory of arbitrage and its consequences, the real difficult part of the story.

In the standard approach to option pricing, that will be called "universal", one derives the consequences of no-arbitrage *before* envisaging the valuation of a given derivative such as an option. As is well-known, no-arbitrage means that if one can design an investment strategy (by taking positions on the existing traded financial securities) whose initial value is equal to zero at the current time, its future cash-flows can not be a positive random variable. No-arbitrage thus excludes the situation in which it is possible to make money with positive probability with a costless investment strategy. The consequences of this seemingly simple no-arbitrage condition turns out to be a very difficult mathematical problem in general because one must show that it is possible to *separate* the set of costless self-financing strategies from the set of positive random variables. Invoking a separation theorem when the conditions are satisfied[1], it is shown that no-arbitrage is equivalent to the existence (but not uniqueness) of a linear functional, interpreted as a pricing probability measure $\mathbb{Q}$, which is such that the discounted value of a financial security such as a stock (the discount rate is the risk-free rate) is a $\mathbb{Q}$-*martingale*, i.e., $\mathbb{E}^{\mathbb{Q}}\left(e^{-rT}S_T|S_0\right) = S_0$. However, once this measure is known (or chosen in case it is not unique), the valuation of a derivative such as a European option is a rather straightforward problem since it reduces to the computation of an expected value. For instance, if one uses a one period binomial model to price a derivative, when the pricing measure $\mathbb{Q} = (q, 1-q)$ is known, one can value any derivative written on the underlying asset such as a stock. In this sense, this general approach to no-arbitrage is *universal* because the pricing measure $\mathbb{Q}$ can be used to price *any* derivative written on the underlying

☆ The authors want to thank two anonymous reviewers for their helpful comments and remarks on this paper. The usual disclaimer obviously apply.

* Corresponding author. Tel.: +330155911010.
E-mail addresses: y.braouezec@ieseg.fr, yann.braouezec@gmail.com (Y. Braouezec), cyril.grunspan@devinci.fr (C. Grunspan).

[1] In continuous time models, no-arbitrage becomes no-free lunch to apply a separation theorem.

http://dx.doi.org/10.1016/j.ejor.2015.08.024
0377-2217/

asset. This approach is presented in most (more mathematically inclined) textbooks such as Bjork (2004), Elliott and Kopp (2005), Lamberton and Lapeyre (2008), Musiela and Rutkowski (1998). When we work with a finite market model (see e.g., Elliot & Kopp 2005 or Pliska 1997), as pioneered by Ritchken (1985) and Taqqu and Willinger (1987), the determination of the option pricing bounds can be done by using linear programming (Ritchken & Kuo, 1988; Musiela & Rutkowski, 1998; King, 2002; Van der Hoek & Elliott, 2006; Antonelli, Mancini, & Pinar, 2013; Camci & Pinar, 2009).

From a practical point of view, the problem is not, in general, to price *any* conceivable derivatives written on an underlying asset such as a stock, but to price a *given* derivative or a small subset of derivatives. Thus, if the difficult part is to derive the consequence of no-arbitrage in general, why not directly focus on the consequences of no-arbitrage for a given option? This is the approach which is followed in the present paper.

We consider here the case of a *given* option (or a linear combination of options with the same maturity), and we show that the determination of the set of arbitrage-free prices reduces to a well-known *convex hull problem*. In particular, the determination of the option pricing bounds does not require to use a pricing measure. Within our approach, no-arbitrage simply means that a certain point, called the forward point, must belong to the convex hull spanned by a set of points, which is either a convex polygon (incomplete market situation) or a segment (complete market situation). For a single period option pricing problem, i.e., with two dates, 0 and $T > 0$, the forward price is defined by the two dimensional vector $(e^{rT}S_0; e^{rT}\Pi_0)$, where S_0 and Π_0 are respectively the stock price and the option price at the current time, and the convex hull is spanned by the set of points $(S_T(\omega); \Pi_T(\omega))$, where $\omega \in \Omega$ is a state of the world (or scenario). An interesting aspect of our approach is that the underlying set of (stock) prices needs not be a finite set. For instance, if this set is a compact subset of the real line, the convex hull (of the graph of the option payoff) is still a convex polygon or a segment, and nothing is changed.[2] What matters is thus not the characteristics of the set of prices, but the *number of vertices* of the convex hull. When this number is equal to two, the option price is unique and the market is complete. On the contrary, when it is greater (or equal) than three, the option price is not unique and the market is incomplete. For European options valued with a continuous time model (pure diffusion, jump-diffusion, infinite activity Levy), things are slightly more complex since the convex hull (of the graph of the option payoff) is not anymore a polygon, but, as we shall see, the European option pricing bounds can still be easily determined.

In a multiperiod model, the determination of the option pricing bounds requires to determine a final convex polygon (possibly a segment) which is obtained via a *sequence of convex hulls*. This (backward induction) process applies both to European and American options with finite maturity and never requires to use a pricing measure. While straightforward, the case of many periods can be tedious from a computational point of view. As a result, we illustrate our geometric approach using a two period model, where all the computation can be easily done without a computer. For simplicity, as in Boyle (1988) or Broadie and Detemple (2004), we consider the case of a trinomial model to value European but also American options. We are able to explicitly compute the pricing bounds as a function of the parameters of the trinomial model. As already said, this entails to determine a sequence of convex hulls, and thus does not make any use of a pricing measure. Since we end up with a convex polygon (possibly a segment) the forward point can be expressed as a convex combination of its vertices. The non-negative coefficients of such a combination, although generally not unique, define a *local system of coordinates with respect to the vertices*, and are called the barycentric coordinates.[3] Let us denote by $\mathbb{Q}$ the barycentric coordinates of the forward point. It becomes elementary to show that the discounted stock price, but also the discounted value of the option, are $\mathbb{Q}$-martingales, which means that the barycentric coordinates $\mathbb{Q}$ can be interpreted as a *martingale measure*. Illustrations are provided for the case of a single period trinomial model for which the barycentric coordinates of a point inside a convex polygon are *unique*, which does of course not mean that the option price is unique. It is important to realize that within our approach, the pricing measure $\mathbb{Q}$ critically depends on the derivative (call, put digital etc.) under consideration and is, in general, not equivalent to the underlying statistical measure $\mathbb{P}$ due to the fact that many points are not relevant. However, it turns out that there are barycentric coordinates (that maximize an entropy function) for which $\mathbb{Q}$ becomes equivalent to $\mathbb{P}$. This more technical material is presented in the last section of this paper in a multi assets framework.

The remaining part of this paper is organized as follows. The first section is devoted to the assumptions. The second and third section are respectively devoted to the determination of the option pricing bounds and to the barycentric coordinates, interpreted as martingale measures. Since we offer a new elementary geometric approach, we made the choice to present the main results as propositions, and all the other elementary results as facts. Twenty facts are presented.

2. Assumptions

Two basic types of financial securities will be considered here, the stock S of a given company, and a (default) risk-free bond that pays a constant interest rate $r > 0$. Throughout this paper, continuous compounding will be assumed. The discount factor thus is e^{-rt} but all the results hold for deterministic interests rates. Throughout the paper, $t = 0$ denotes the current time and $T > 0$ the maturity of the derivative contract written on the underlying asset S.

Dupačovà, Consigli, and Wallace (2000) introduced a framework intended for multistage stochastic programs (see also King, 2002 for an option pricing application) called scenario tree which encompasses recombining and non-recombining trees. It is used by Antonelli et al. (2013), Camci and Pinar (2009) and King (2002) for option pricing and they all consider the case in which the tree is non-recombining, i.e., in which each node at each time has a *unique* parent. In this paper, we consider the case in which the tree is recombining, i.e., in which each node may have more than one parent. This allows us to consider for instance a multiplicative multinomial stochastic process but it should be clear that our methodology holds indeed for any type of trees, recombining or not.

Let $\mathcal{T} = \{1, 2, \ldots, T\}$ be a set of dates, with $T \geq 1$, and let $(m_{i,t})_{i=1}^N$ be $N \geq 2$ *ordered* positive numbers for all t, where $m_{1,t} = d_t$ and $m_{N,t} = u_t$. Let ζ_s be a random variable whose realization denoted m_s will be revealed at time s. Let $\Xi_s := \{d_s, m_{2,s}, \ldots, m_{N-1,s}, u_s\}$ be the set of possible realizations[4] of ζ_s for all $s \in \mathcal{T}$. The multiplicative multinomial (possibly time-dependent, i.e., Ξ may depend on time s) stochastic process is modeled, for $t \geq 1$, by setting

$$S_t = S_0 \prod_{s=1}^{t} \zeta_s \qquad t \leq T \tag{1}$$

where S_0 is the initial condition, i.e., the observed stock price at time $t = 0$. When $N = 2$, that is, when the random variable ζ_s takes only

[2] We make the implicit assumption that $\Pi_T(\omega)$ is finite for each ω.

[3] Cartesian coordinates is a system of coordinates for the representation of a point in an n-dimensional space in terms of its distance, measured along a set of mutually axes, from a given origin. On the other hand, barycentric coordinates locate points relative to a set of existing points rather than an origin. For this reason, they are called local coordinates. See for instance the textbook of Vince (2010) and more specifically chapter 11 devoted to barycentric coordinates.

[4] We mean the support of the underlying probability measure $\mathbb{P}$.