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An explicitly solvable Heston model with stochastic interest rate

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ABSTRACT

This paper deals with a variation of the Heston hybrid model with stochastic interest rate illustrated in Grzelak and Oosterlee (2011). This variation leads to a multi-factor Heston model where one factor is the stochastic interest rate. Specifically, the dynamics of the asset price is described through two stochastic factors: one related to the stochastic volatility and the other to the stochastic interest rate. The proposed model has the advantage of being analytically tractable while preserving the good features of the Heston hybrid model in Grzelak and Oosterlee (2011) and of the multi-factor Heston model in Christoffersen et al. (2009). The analytical treatment is based on an appropriate parametrization of the probability density function which allows us to compute explicitly relevant integrals which define option pricing and moment formulas. The moments and mixed moments of the asset price and log-price variables are given by elementary formulas which do not involve integrals. A procedure to estimate the model parameters is proposed and validated using three different data-sets: the prices of call and put options on the U.S. S&P 500 index, the values of the Credit Agricole index linked policy, Azione Più Capitale Garantito Em.64., and the U.S. three-month, two and ten year government bond yields. The empirical analysis shows that the stochastic interest rate plays a crucial role as a volatility factor and provides a multi-factor model that outperforms the Heston model in pricing options. This model can also provide insights into the relationship between short and long term bond yields.

1. Introduction

The pricing of derivative products is probably one of the most challenging topics in modern financial theory. The modern fixed income market includes not only bonds but also derivative securities sensitive to interest rates. In this paper we consider a modified version of the hybrid model illustrated in Grzelak and Oosterlee (2011). This model is described by a system of stochastic differential equations (SDEs) which combines different models for equity, interest rate and volatility in order to efficiently price European vanilla call and put options with short and long maturities.

Specifically, we focus on the model which combines the Heston model 1993 for equity and its volatility and the Cox, Ingersoll, and Ross (1985) (CIR) model for interest rate. Roughly speaking, the proposed model can be interpreted as the Heston multi-factor model introduced by Christoffersen, Heston, and Jacob (2009) where one volatility factor is the stochastic interest rate. Indeed, a multi-factor Heston-like stochastic volatility model capable of explaining some stylized facts on interest rate volatility and of forecasting bond yields with different maturities has been introduced by Trolle and Schwartz (2009). This model describes the yield using a stochastic differential equation where the drift and the coefficients of the stochastic variances are dependent on time and maturity date.

Furthermore, Cieslak and Povala (2014) propose the joint use of a short term yield with another stochastic factor as an efficient tool to explain the volatility of the yields. Motivated by the results of these papers, the hybrid model proposed here has been applied to option pricing (see Sections 4.2 and 4.3) and, at a very preliminary stage, to bond yield analysis (see Section 4.4).

1.1. Literature review

Recent literature motivates the use of hybrid SDE models due to the empirical evidence that the asset volatility and the interest rate are not constant over time. Indeed, the relaxation of the constant volatility assumption in time continuous stochastic volatility models goes back to the end of 1980s with (Ball & Roma, 1994; Heston, 1993; Hull & White, 1988; Stein & Stein, 1991).

The Heston model is one of the most celebrated models because it allows for closed-form formulas for option pricing. In fact, this model accurately describes the asset price behavior when the

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assumption of constant interest rate is realistic and the volatility is not affected by abrupt oscillations. Improvements of this model can be found in the recent literature with the models of Christoffersen et al. (2009), Fatone, Mariani, Recchioni, and Zirilli (2009), Fatone, Mariani, Recchioni, and Zirilli (2013), Wong and Lo (2009), Date and Islyaev (2015), Islyaev and Date (2015), Pun, Chung, and Wong (2015). These models are modified versions of the Heston model which are able to better capture the price volatility dynamics in order to efficiently solve option pricing problems.

The relaxation of the constant interest rate assumption can be found in the literature of the last decade. Far from being exhaustive, we cite the papers of Chiarella and Kwon (2003), Trolle and Schwartz (2009), Andersen and Benzoni (2010), Christensen, Diebold, and Rudebush (2011), Moreno and Platania (2015) which show that stochastic interest rates should be used in order to capture the bond yield behavior.

In line with the attempt to deal with stochastic interest rates and stochastic volatility(ies) several hybrid SDE models have been introduced since 2000. In fact, Zhu (2000) introduces a model capable of generating a skew pattern for the equity using a stochastic interest rate not correlated to the equity. Later, Andreasen (2007) generalizes the Zhu model using the Heston stochastic volatility model and an indirect correlation between the equity and the interest rate process. Ahlip (2008) studies a model for the spot FX (Foreign Exchange) rate with stochastic volatility and stochastic domestic as well as foreign rates. Specifically, these rates are modeled by the Ornstein–Uhlenbbeck process and the volatility is modeled by a mean-reverting Ornstein–Uhlenbeck process correlated with the spot FX rate. Ahlip also derives an analytical formula for the price of European call options on the spot FX rate. Grzelak, Oosterlee, and Van Weeren (2012) propose the so called Schöbel–Zhu–Hull–White hybrid model. This is an affine model whose analytical treatment is shown by Grzelak et al. (2012) following the approach proposed by Duffie, Pan, and Singleton (2000). However, as highlighted in Grzelak and Oosterlee (2011), the model allows for negative volatility and interest rate. In order to overcome this problem they propose the use of a Cox–Ingersoll–Ross (CIR) process to describe the variance and interest rate processes. Local volatility models have also been extended to deal with stochastic interest rates. For example, Deelstra and Rayee (2012) propose a three factor pricing model with local volatility and domestic and foreign interest rates modeled by the Hull and White (HW) model 1993. In line with the latter, Benhamou, Gobet, and Miri (2012) provide analytical formulas for European option prices when the underlying asset is described by a local volatility model with stochastic rates.

1.2. Description of the results

As previously mentioned, this paper focuses on a modified version of the hybrid Heston-CIR model illustrated by Grzelak and Oosterlee (2011). However, the analytical treatment proposed here extends to the hybrid Heston-Hull and White model. Our contribution is threefold.

Firstly, we modify the hybrid SDE model of Grzelak and Oosterlee (2011) in order to preserve the affine structure and to permit a "direct" correlation between the equity and the interest rate. As highlighted by Grzelak and Oosterlee (2011) this correlation plays a fundamental role in getting a good match between the observed and the theoretical option prices.

Secondly, we present the analytical treatment of the model. Specifically, we derive a parametrization of the probability density function of the stochastic process by solving the backward Kolmogorov equation using some ideas illustrated in Fatone et al. (2009, 2013).

This parametrization allows us to express the prices of European call and put options as one dimensional integrals and to derive elementary formulas for the moments and mixed moments of the asset price variable as well as the moments of the log-price variable. These elementary formulas which do not involve integrals show that the existence of bounded moments of the price variable depends on the value of the correlation coefficients. This finding coheres with the results obtained by Lions and Musiela (2007) as well as Andersen and Piterbarg (2007) on the explosion of moments of some well known probability distributions. As a byproduct of this analytical treatment we obtain an efficient approximation of the stochastic integral appearing in the discount factor (i.e. an efficient approximation of the zero-coupon bond). This approximation is suggested by the explicit formula of the zero-coupon bond in the CIR model (see Eq. (29)) and permits us to approximate the option prices as one dimensional integrals of explicitly known elementary functions.

Thirdly, we calibrate the model in order to measure its performance in interpreting real data and forecasting European call and put option prices (see the empirical analysis on the U.S. S&P 500 index options in Section 4.2 and on the Credit Agricole pure endowment policy in Section 4.3) and in predicting the trend of the two and ten-year U.S. government bond yields (see Section 4.4).

The calibration procedure of the empirical analysis of Sections 4.2 and 4.3 is based on the solution of a nonlinear constrained optimization problem whose objective function measures the relative squared difference between the observed and theoretical call and/or put option prices while the calibration of Section 4.4 is based on the maximum likelihood approach. Numerical simulations show that the approximate formulas for the discount factor and option prices work satisfactorily for any maturities (see Section 4.1). Moreover, the empirical study conducted using real data shows that the model is capable of fitting and predicting satisfactorily call and put option prices using only one set of model parameters obtained from the calibration procedure (empirical analysis of call and put options on U.S. S&P 500 index). In addition, it is capable of forecasting values of long-term products (Credit Agricole pure endowment policy) and of interpreting the relationship between short and long term bond yields.

The empirical analysis regarding the pure endowment policy and the two/ten-year U.S. government bonds show the crucial role played by the stochastic interest rate factor when long maturities are considered. However, we show that the interest rate is a significant factor even in the case of European options on the U.S. S&P 500 index. To do this we use a two-stage calibration. In the first stage we estimate the parameters of the CIR model by using the U.S. three month government bond yields and then we estimate the remaining parameters by using option prices. The hybrid model calibrated in this way still outperforms the Heston model highlighting that the stochastic interest rate is a crucial factor (see Section 4.2).

1.3. Outline of the paper

The paper is organized as follows. In Section 2 we describe the hybrid SDE model and illustrate the main relevant formulas. In Section 3 we propose formulas to approximate the European vanilla call and put option prices as one-dimensional integrals of explicitly known functions. In Section 4 we illustrate some experiments involving the moments of the price variable and the zero coupon bond formula (Section 4.1). Furthermore, we estimate the model parameters implied by the option prices by solving constrained optimization problems. We use the daily prices of European vanilla call and put options on the U.S. S&P 500 index from April 2, 2012 to July 2, 2012 in Section 4.2, the weekly values of the Credit Agricole pure endowment policy from April 4, 2012 to April 13, 2015 in Section 4.3. In Section 4.4 the daily data of the three-month, two(ten)-year U.S.