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Discrete Optimization

Graphical exploration of the weight space in three-objective mixed integer linear programs

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ABSTRACT

In this paper we address the computation of indifference regions in the weight space for multiobjective integer and mixed-integer linear programming problems and the graphical exploration of this type of information for three-objective problems. We present a procedure to compute a subset of the indifference region associated with a supported nondominated solution obtained by the weighted-sum scalarization. Based on the properties of these regions and their graphical representation for problems with up to three objective functions, we propose an algorithm to compute all extreme supported nondominated solutions adjacent to a given solution and another one to compute all extreme supported nondominated solutions to a three-objective problem. The latter is suitable to characterize solutions in delimited nondominated areas or to be used as a final exploration phase. A computer implementation is also presented.

1. Introduction

A multiobjective integer or mixed-integer linear programming (MOMILP) problem with $p \geq 2$ objective functions can be written as:

$$\left.\begin{array}{l} \max\ z_1 = f_1(x) = c^1 x \\ \quad \dots \\ \max\ z_p = f_p(x) = c^p x \end{array}\right\} \text{Max } z = f(x) = Cx$$

$$\text{s.t.} \quad x \in X = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0,\ x_j \in \mathbb{N}_0,\ j \in I\}$$

where A is the $m \times n$ technological coefficients matrix, being all constraints transformed into equations by introducing appropriate slack or surplus variables, and $b \in \mathbb{R}^m$ is the right-hand-side vector. $I \subseteq \{1, \dots, n\}$, $I \neq \emptyset$ is the set of indices of the integer variables, with n the total number of variables (decision variables plus slack/surplus variables). It is assumed that X is bounded and nonempty. C is the $p \times n$ objective matrix whose rows are the vectors $c^k \in \mathbb{R}^n$, $k = 1, \dots, p$.

If all decision variables are integer, then the multiobjective problem is pure integer (MOILP), which is a special case of the multiobjective mixed-integer case. In what follows we will denote by MOMILP the general case, in which integrality constraints are imposed on all or a subset of the decision variables.

A feasible solution $x' \in X$ is *efficient* if and only if there is no other solution $x \in X$ such that $f_k(x) \geq f_k(x')$ for all $k = 1, \dots, p$ and $f_k(x) > f_k(x')$ for at least one k. Let X_E denote the set of all efficient solutions.

Let $Z \subset \mathbb{R}^p$ be the image of the feasible region X in the objective space such that $Z = \{z \in \mathbb{R}^p: z = Cx,\ x \in X\}$. If $x' \in X$ is efficient, $z' = f(x') = Cx'$ is a *nondominated* criterion point. Let Z_{ND} be the set of all nondominated points, $Z_{ND} = \{z' \in Z : z' = Cx', x' \in X_E\}$.

An important concept in MOMILP is the distinction between *supported* and *unsupported* nondominated/efficient solutions.

A nondominated point $z' \in Z_{ND}$ is *supported* if it is located on the boundary of the convex hull of Z (*conv Z*). An *unsupported* nondominated point is located in the interior of *conv Z* (it is dominated by some convex combination of supported nondominated points). A supported (unsupported) nondominated point corresponds to a supported (unsupported) efficient solution.

We can further distinguish two types of supported nondominated points:

(a) extreme supported nondominated points $z \in Z_{ND}$, which are vertices of *conv Z*; we will denote these nondominated points/efficient solutions as ESND solutions;
(b) non-extreme supported nondominated points, which are located in the relative interior of a face of *conv Z*.

Let Z_{ESND} denote the set of all ESND points/solutions in the objective space and X_{ESND} the corresponding set in the decision space.

Supported nondominated solutions are optimal solutions to the weighted-sum scalarization program (P_λ) for some weight vector $\lambda \in \Lambda = \{\lambda \in \mathbb{R}^p : \lambda_k > 0, k = 1, \dots, p, \sum_{k=1}^{p} \lambda_k = 1\}$:

$$\max z_\lambda = \sum_{k=1}^{p} \lambda_k f_k(x) = \lambda C x \qquad (\text{P}_\lambda)$$
$$\text{s.t.} \quad x \in X$$

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Λ is usually called the *weight space* and the set of weight vectors that lead to the same nondominated solution is referred to as an *indifference region* in the weight space. The weights of the objective functions are the parameters in the (P_λ) scalarization program and the variation of parameters enables to attain different supported nondominated solutions. However, there are multiple parameter values that lead to the same solution, i.e. an indifference set on the parameter's space (weight space) can be defined for each supported nondominated solution.

The set Λ can be decomposed into subsets $\Lambda(z')$, $\forall z' \in Z_{ND}$ such that z' is supported, where $\Lambda(z')$ denotes the indifference region of z' in the weight space. It represents the set of weight vectors λ that lead to z' through the optimization of (P_λ), i.e., $\Lambda(z') = \{\lambda \in \Lambda: \lambda z' \geq \lambda z, \forall z \in Z_{ND}\}$. Indifference regions in the weight space are convex polytopes (Przybylski, Gandibleux, & Ehrgott, 2010).

The optimization of (P_λ) using the branch-and-bound method yields (at least) an ESND solution. If there are alternative optima, a further exploration of the branch-and-bound tree allows for computing non-extreme supported nondominated solutions. However, unsupported nondominated solutions are never obtained through (P_λ) even if a complete parameterization is attempted and all alternative solutions for a given $\lambda \in \Lambda$ are analysed. The ESND points allow for the whole decomposition of the weight space into subsets $\Lambda(z')$, $z' \in Z_{ESND}$, because these, and only these points $z' \in Z_{ESND}$, correspond to indifference regions $\Lambda(z')$ of dimension $p-1$ (the dimension of Λ) of a MOMILP problem. Therefore, $\Lambda = \bigcup_{z' \in Z_{ESND}} \Lambda(z')$. Non-extreme supported nondominated points are associated with indifference regions of lower dimension resulting from the intersection of the regions of ESND points (these properties can be found in Przybylski et al. (2010)).

Although the supported nondominated solutions (or even only the ESND solutions) constitute a subset of all nondominated solutions of the problem, they can provide important insights about the whole nondominated set because they are on the boundary (and the ESND are the vertices) of the convex hull of all nondominated points (Özpeynirci & Köksalan, 2010). Indifference regions in the weight space also constitute useful information for the decision maker. He/she may be indifferent to all weight combinations inside one region because they give rise to the same nondominated point.

An interactive graphical exploration of the weight space in multiobjective linear programming (MOLP) problems with three objective functions has been proposed in the TRIMAP method by Clímaco and Antunes (1987). The use of the weight space as a valuable means to gather information obtained from different interactive methods, and its graphical representation to present the information to the decision maker, has likewise been considered in other interactive MOLP computational tools (Alves, Antunes, & Clímaco, 2015; Antunes, Alves, Silva, & Clímaco, 1992). Also considering MOLP problems, Benson and Sun (2002) proposed a weight set decomposition algorithm to generate all extreme nondominated points.

The computation of indifference regions in other parameter spaces has also been addressed. Costa and Clímaco (1999) related reference points (using achievement scalarizing functions) and weights in MOLP and defined indifference regions on the reference point space. Alves and Clímaco (2001) analysed the shape of indifference regions in the reference point space (in general, non-convex regions) for all-integer MOILP problems and proposed an approach to define indifference sets of reference points as long as a *directional search* procedure (Alves & Clímaco, 2000) is performed.

Concerning MOMILP problems, Przybylski et al. (2010) and Özpeynirci and Köksalan (2010) have exploited the weight space to design algorithms intended to generate all ESND points. These two approaches are reviewed in the next section.

In the present work we focus on ESND solutions of MOMILP problems and the exploration of their indifference regions in the weight space. We propose an approach that is able to compute a subset of an indifference region using the branch-and-bound tree that solved the weighted-sum scalarizing program (P_λ) for a given weight vector $\lambda \in \Lambda$. Acting alone, this approach rarely calculates the entire indifference region for the corresponding ESND solution, and the obtained sub-region may be much smaller than the full region. However, indifference regions can be iteratively enlarged using some properties, namely convexity. Accordingly, we have developed a procedure that merges and expands sub-regions by building the convex hull of joined or disjoined sub-regions of the same solution. An indifference region can be enlarged not only from a merging process but also as a result of properties that relate adjacent regions of different solutions. We explore these properties for three objective problems, proposing an algorithm to compute all ESND solutions adjacent (in the weight space sense) to a known ESND solution or even to compute all ESND solutions of a three-objective problem. These features can naturally be applied to problems with two objective functions, but we will omit this case herein as it is straightforward. We also present a computer implementation in which the indifference regions are graphically depicted.

The rest of the paper is organized as follows. In Section 2 the related work is reviewed. Section 3 introduces the technique to compute an indifference sub-region for an ESND solution and an illustrative example is shown. Section 4 gives the main principles to explore the weight space in MOMILP problems with three objective functions and describes the algorithms to compute the ESND solutions adjacent to a given solution and to compute all ESND solutions. Section 5 presents an overview of the computational implementation, illustrating the previous features using two examples, and presents computational experiments. The paper ends with some concluding remarks and future work in Section 6.

2. Related work

Przybylski et al. (2010) proposed a recursive algorithm for finding Z_{ESND} which is based on the following idea: in each iteration, the weight space is completely decomposed with the solutions computed so far and the common facets of the regions are explored in order to compute new solutions and update the weight space decomposition. Let $S \subseteq Z_{ESND}$ denote the set of ESND points known at a given iteration. For each $z' \subseteq S$ a *super-region* $\Lambda^+(z')$ is defined such that $\Lambda^+(z') = \{\lambda \in \Lambda : \lambda z' \geq \lambda z, \forall z \in S\}$ Then, the algorithm searches for new nondominated points at the boundaries of $\Lambda^+(z')$. For instance in three-objective problems, if $z', z'' \in S$ are adjacent in the current weight space decomposition, then $\Lambda^+(z') \cap \Lambda^+(z'')$ is a line segment; suppose that λ^1 and λ^2 are the extreme points of this edge; the algorithm investigates the edge by computing all the ESND points of the following bi-objective problem: $\max\{(f_1'(x), f_2'(x)) = (\lambda^1 Cx, \lambda^2 Cx) : x \in X\}$. Thus, a recursive algorithm is developed. As new ESND points are added to S, the weight space decomposition is updated until all elements of Z_{ESND} have been found. At the end of the algorithm, the regions $\Lambda^+(z')$ are the real ones, i.e. $\Lambda^+(z') = \Lambda(z')$. The authors have further proved that a suitable initialization of the algorithm must contain the nondominated extreme points that optimize individually each objective function. This initialization enables to only explore the facets of each $\Lambda^+(z')$ that are not located on the boundary of Λ.

Özpeynirci and Köksalan (2010) proposed another algorithm with the same purpose of finding all points of Z_{ESND}. This algorithm does not use recursion, but it has a strong combinatorial component. The basic idea consists in introducing p dummy points in the objective space, $Z_m = \{m^k = Me^k, k = 1,\ldots,p\}$ where M is a large positive constant and e^k is the k^{th} unit vector. These points are infeasible and nondominated with respect to all points of Z_{ESND}. They have such characteristics that their indifference regions in the weight space touch all the boundary of Λ (where one of the weights is close to zero). The points m^k are incorporated into the search, which turns to be on