European Journal of Operational Research 246 (2015) 721–729

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Continuous Optimization

An inexact proximal method for quasiconvex minimization

E. A. Papa Quiroz[a,b,*], L. Mallma Ramirez[c], P. R. Oliveira[a]

[a] Federal University of Rio de Janeiro, Brazil
[b] Mayor de San Marcos National University, Perú
[c] Callao National University, Perú

ARTICLE INFO

Article history:
Received 30 November 2013
Accepted 17 May 2015
Available online 22 May 2015

Keywords:
Computing science
Global optimization
Nonlinear programming
Proximal point methods
Quasiconvex minimization

ABSTRACT

In this paper we propose an inexact proximal point method to solve constrained minimization problems with locally Lipschitz quasiconvex objective functions. Assuming that the function is also bounded from below, lower semicontinuous and using proximal distances, we show that the sequence generated for the method converges to a stationary point of the problem.

1. Introduction

It is well known that the class of proximal point algorithm (PPA) is one of the most studied methods for finding zeros of maximal monotone operators and in particular to solve convex optimization problems, see Auslender and Teboulle (2006), Burachik and Iusem (1998), Burachik and Scheimberg (2000), Chen and Teboulle (1993), Kiwiel (1997), Rockafellar (1976).

In the last decades a great interest has emerged to extend the PPA for non monotone variational inequalities and non convex minimization problems not only for extending the convergence theory but by several applications in diverse science and engineering areas, see for example the works of Attouch and Bolte (2009); Attouch, Bolte, and Svaiter (2013); Kaplan and Tichatschke (1998); Pennanen (2002); Chen and Pan (2008).

In particular the class of quasiconvex minimization problems has been receiving attention from many research works due to the broad range of applications in location theory, Gromicho (1998), Fractional programming and specially in economic theory, see for example Takayama (1995) and Mas-Colell, Whinston, and Green (1995). Some related papers are the following: Goudou and Munier (2009), Souza, Oliveira, da Cruz Neto, and Soubeyran (2010), Chen and Pan (2008), Cunha, da Cruz Neto, and Oliveira, (2010), Pan and Chen (2007), Papa Quiroz and Oliveira (2012), Langenberg and Tichatschke (2012), Brito, da Cruz Neto, Lopez, and Oliveira (2012).

In this paper we are interested in extending the global convergence of an inexact proximal point method to minimize a quasiconvex function constrained on a nonempty closed convex set, that is,

$$\min\{f(x) : x \in \bar{C}\}, \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a quasiconvex function, C is open convex set on the euclidean space $\mathbb{R}^n$ and $\bar{C}$ is the closure of C. Obviously, if $C = \mathbb{R}^n$ we obtain the unconstrained minimization problem. Some convergence results have been recently obtained for some research works:

Attouch and Teboulle (2004), with a regularized Lotka–Volterra dynamical system, have proved the convergence of the continuous method to a point which belongs to certain set which contains the set of optimal points; see also Alvarez, Bolte, and Brahic (2004), that treats a general class of dynamical systems that includes the one of Attouch and Teboulle.

Souza et al. (2010), Cunha et al. (2010), Chen and Pan (2008) and Pan and Chen (2007) studied the iteration

$$x^k \in \arg\min_{x\in\bar{C}}\{f(x) + \lambda_k d(x, x^{k-1})\}, \tag{1.2}$$

where $\bar{C} = \mathbb{R}^n_+$, d is a certain distance to force the iterates x^k to stay in C. Some examples of d based on the literature are the class of Bregman, $\varphi-$divergence and second order homogeneous distances. Under the assumption that f is bounded from below, $\mathrm{dom}(f) \cap \mathbb{R}^n_+ \neq \emptyset$, the proximal parameter is bounded and assuming that f is differentiable, these research works obtained the global convergence of the method to a *KKT* point of (1.1). Furthermore, the sequence generated converges to a solution of the problem, should it exist, if the proximal parameters approach to zero.

* Corresponding author at: Federal University of Rio de Janeiro Brazil. Tel.: +51 1 5685537.
E-mail addresses: erikpapa@gmail.com (E.A. Papa Quiroz), lenninmr@gmail.com (L. Mallma Ramirez), poliveir@cos.ufrj.br (P.R. Oliveira).

http://dx.doi.org/10.1016/j.ejor.2015.05.041
0377-2217/

Brito et al. (2012), proposed an interior proximal algorithm inspired by the logarithmic-quadratic proximal method for linearly constrained quasiconvex minimization problems. For that method, they proved the global convergence when the proximal parameters go to zero. The latter assumption could be dropped when the function is assumed to be pseudoconvex.

Langenberg and Tichatschke (2012), motivated from the work of Kaplan and Tichatschke (1998), studied the iteration (1.2) when C is an arbitrary open convex set and d is a Bregman distance. Assuming that f is locally Lipschitz and using the Clarke subdifferential, the authors proved the global convergence of the method to a critical point of (1.1).

The above works although important have some disadvantages:

- The main difficulty in extending the proximal method for nonconvex function, which was observed by Kaplan and Tichatschke (1998) and Langenberg and Tichatschke (2012), is that due to the nonconvexity of f the subproblems of (1.2) may not be convex and thus, from a practical point of view, we may obtain that minimization subproblems may be as hard to solve globally as the original one due to the existence of multiple isolated local minimizers. These authors have proved, under some appropriate conditions and chose a sufficiently large regularization parameters, the strong convexity of the proximal subproblems (and thus efficiently solvable subproblems) for a class of non convex functions, see Theorem 2 and Theorem 5 of Kaplan and Tichatschke (1998) and Langenberg and Tichatschke (2012) respectively. However, the above property is not true in general for arbitrary quasiconvex functions. So, we believe that a basic idea is to weaken the condition of minimizing a strongly convex regularization function by another one such that in each iteration we may use local information of the subproblems. If the regularized function is, for example, locally strongly convex we will have, in certain sense, efficiency in solving the subproblems. This motivates the following question: Is it possible to introduce a local stationary iteration that makes much more sense that the previously considered (1.2) for dealing with nonconvex problems?
- In Rockafellar (1976) it is shown that in some cases let the proximal parameter converges to zero, although the regularizing effect vanishes, provides superlinear convergence of the algorithm in the convex case. Motivated by this fact, Brito et al. (2012), Souza et al. (2010), Cunha et al. (2010), Chen and Pan (2008) and Pan and Chen (2007) have been proved the convergence of the proximal method to an optimal point when the parameters converges to zero. On the other hand, when the proximal parameters are sufficiently greater than zero but bounded from above, Langenberg and Tichatschke (2012), see Theorem 9 of that work, proved the convergence to a stationary point which may be in the worst case a saddle point (observe that stationary point or critical point point does not necessarily a global nor local minimum point). This motivates the following question: is it possible to obtain a convergence theory to an optimal point when the proximal parameters are bounded from above?
- Despite the fact that the proximal point method is not practical in its exact version, several works, for the convex case, have been shown that it is possible to obtain implementable algorithms with good convergence properties, see for example Alvarez, López, and Ramírez (2010), Liu, Sung, and Toh (2012), Santos and RC (2014). In our case, for a computational implementation of the proximal point algorithm for the quasiconvex case it is needed to solve the iteration (1.2) using a local optimization algorithm, which only provides an approximate solution. Thus it is important to consider inexact methods. Therefore, from the computational point of view, is it possible to introduce a inexact proximal method to solve (1.2) and prove the convergence of the iteration?

In this paper, motivated by a recently work of Papa Quiroz and Oliveira (2012), we answer the questions of the first and third bullet points and partially we answer the question of the second bullet one, see Subsection 4.3, proposing the following proximal method: given $x^{k-1} \in C$, find $x^k \in C$ and $g^k \in \partial^\circ f(x^k)$ such that

$$\left\|x^{k-1} - x^k - e^k\right\| \le max\left\{\left\|e^k\right\|, \left\|x^k - x^{k-1}\right\|\right\} \tag{1.3}$$

where

$$e^k = g^k + \lambda_k \nabla_1 d(x^k, x^{k-1}) \tag{1.4}$$

with ∂° is the Clarke subdifferential and d is a proximal distance, see Sections 2 and 3 respectively.

The condition (1.3) has been motivated from the work of Humes and Silva (2005) and Solodov and Svaiter (1999), where they considered the following criterion:

$$||g^k + (x^k - x^{k-1})|| \le \sigma \max\{||g^k||, ||x^k - x^{k-1}||\}$$

with $\sigma \in (0, 1]$. Observe that the above condition and (1.3) are different and so we may conclude that (1.3) is new in proximal point methods even for the convex case. When $e^k = 0$ in (1.4) and $C = \mathbb{R}^n_{++}$ we obtain the exact version studied by Papa Quiroz and Oliveira (2012) for the nonnegative orthant.

Observe also that the conditions (1.3)-(1.4) are more practical than (1.2), where a global minimum point is required in each iteration, and thus more practical than the works of Souza et al. (2010), Cunha et al. (2010), Chen and Pan (2008), Pan and Chen (2007) and Langenberg and Tichatschke (2012). Therefore in our opinion the local stationary iterations (1.3)–(1.4) makes much more sense than the previously considered (1.2) for dealing with nonconvex problems.

Under the assumption that f is proper, lower semicontinuous, locally Lipschitz and bounded from below on $\bar{C}$ and using a class of proximal distance we will prove that $\{x^k\}$ is well defined and if, in addition, f is quasiconvex it will be proved that $\{f(x^k)\}$ is decreasing and $\{x^k\}$ converges to some point of $U_+ := \{x \in \bar{C} : f(x) \le \inf_{j\ge 0} f(x^j)\}$, assumed nonempty. Then, under the additional conditions that the proximal parameter $\{\lambda_k\}$ is bounded from above and

$$\sum_{k=1}^{+\infty} \frac{\left\|e^k\right\|}{\lambda_k} < +\infty \tag{1.5}$$

$$\sum_{k=1}^{+\infty} \frac{\left|\langle e^k, x^k\rangle\right|}{\lambda_k} < +\infty, \tag{1.6}$$

we obtain that the sequence $\{x^k\}$ converges to a stationary point of the problem. Observe that the above conditions (1.5)–(1.6) have been used in convex proximal methods, see for example Auslender, Teboulle, and Ben-Tiba (1999), Kaplan and Tichatschke (2004), Eckstein (1998), Xu, Bingsheng, and Xiaoming (2006), Solodov and Svaiter (2000). We also get to rid the assumption (1.6) for a class of induced proximal distances which includes Bregman distances given by the standard entropy kernel and all strongly convex Bregman functions.

The paper is organized as follows: In Section 2 we give some basic results on quasiconvex theory and Clarke subdifferential of locally Lipschitz functions. In Section 3 we introduce the class of proximal distances that we will use along the paper. In Section 4 we present the inexact algorithm for solving minimization problems with quasiconvex functions and analyze its convergence properties. Finally, in Section 5 we give our conclusions.

2. Basic results

Throughout this paper $\mathbb{R}^n$ is the Euclidean space endowed with the canonical inner product $\langle , \rangle$ and the norm of x given by $||x|| := \langle x, x\rangle^{1/2}$. Given $X \subset \mathbb{R}^n$ we denote $bd(X)$ and $\bar{X}$ the boundary and closure