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Short Communication

Risk pricing in a non-expected utility framework

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ABSTRACT

Risk prices are calculated as the certainty equivalents of risky assets, using a recently developed non-expected utility (non-EU) approach to quantitative risk assessment. The present formalism for the pricing of risk is computationally simple, realistic in the sense of behavioural economics and straightforward to apply in operational research and risk and decision analyses.

1. Introduction

The intrinsic price of a risky asset of random monetary value X is the certainty equivalent c_p of the probability distribution $p(X)$. The term “intrinsic” refers to risk quantification within a given accounting system rather than to risk prices extrinsically determined by the market for the asset. In the framework of utility theory, the certainty equivalent of a risk p is implicitly defined by the requirement

$$U(p) = U(c_p), \quad p \in P, \tag{1}$$

where U is a real-valued utility functional defined on a space P of probability distributions and c_p, $c_p \in P$, is the degenerate risk which gives the result $X = c_p$ with certainty. In Eq. (1) and throughout this research note, we do not accordingly distinguish between real numbers x and degenerate probability distributions which give $X = x$ with certainty. We also abbreviate $p(X = x)$ as $p(x)$, as is usual elsewhere.

Procedures to specify certainty equivalents are useful in theoretical and applied risk research. They can help to rank-order risk preferences in simple and consistent ways, to determine intrinsic risk prices in finance and insurance applications and to provide experimental tests of theories of utility (Becker, DeGroot, & Marschak, 1964; Farquhar, 1984; Denuit, Dhaene, Goovaerts, Kaas, & Laeven, 2006). But within the theoretical frameworks of EU and non-EU theory, they rarely admit explicit solutions of Eq. (1) for c_p (Denuit et al., 2006). Alternatively, to solve Eq. (1) for c_p by numerical approximation, the utility and probability weighting functions on which representations of U are typically based must often be determined empirically (for a compilation of the extensive literature, see van de Kuilen & Wakker, 2011) or chosen *ad hoc* as parametric functions (Stott, 2006) and then fitted to the experimental data. The latter step demands, in addition, considerable methodological effort (for review, see Abdellaoui, Bleichrodt, & L’Haridon, 2008). As for general methodological foundations, recent experimental results and reference to the literature on risk-pricing behaviour, see Blavatskyy and Köhler (2009, 2011).

In the following, Eq. (1) will be solved for c_p within a recently developed axiomatic framework of *status quo* dependent risky choice involving a non-EU utility functional U. U accommodates systematic violations of EU theory of various kinds observed in risky choice experiments (Geiger, 2008, 2012). It will be shown that c_p possesses an explicit representation solely involving the cumulative probabilities of gain and loss associated with p and a few exogenous parameters that are not related to p. This result will then be extended to utility preferences for multivariate risks. Applications to recent experimental results on risk pricing behaviour will also be indicated. The present formalism for the intrinsic pricing of risk may thus be of theoretical and practical use in wide areas of operational research and risk management.

2. The analytic framework

We consider a convex set P of simple probability distributions p defined on a compact real interval I. A person’s attitude towards a given risk p is assumed to be governed, besides by p, by his or her neutral reference point x_0, $x_0 \in I$, *status quo* risk s, $s \in P$, and relative persistence ε of p in the presence of s, that is, overall probability ε of $T_p > T_s$, where T_p and T_s are the uncertain times to resolution of p and s. Accordingly, $\varepsilon(T_p > T_s)$ generally varies with p for given *status*

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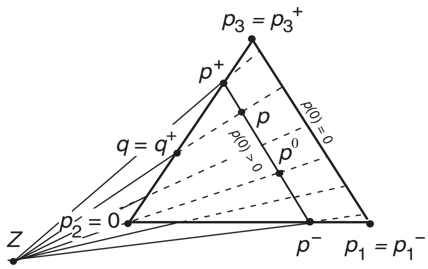

Fig. 1. Indifference lines (dashed) in a probability triangle Δ. The "fanning out" of the indifference lines is familiar from systematic violations of EU theory in risky choice experiments (Starmer, 2000). The risk p with $p \neq p^{\pm}$, $p(0) > 0$, is indifferent to the "pure chance" $q = q^+$.

quo s.[1] To avoid trivial cases, s is non-degenerate and involves at least some chance of gain $s(x) > 0$ for some $x > x_0$, $x \in I$, and some risk of loss $s(x') > 0$ for some $x' < x_0$, $x' \in I$. In applications of the formalism, the parameters x_0, ε and s will normally be measurable or can be estimated with some confidence by standard statistical methods (*status quo* risk) and multivariate survival, or hazard rate, analysis (probability $\varepsilon(T_p > T_s)$). Without loss of generality, x_0 is normalised to $x_0 = 0$. As an option of choice, to stay in one's *status quo* (i.e., choose s given s) amounts to selecting the degenerate risk 0 given s, that is, adding nothing to s with certainty. Hence, $c_s = 0$. As for the axiomatic foundations and for more detailed explanations of the approach, see Geiger (2002, 2008).

To state the basic equations of utility preference for which Eq. (1) is to be solved, the following variables must be defined (Fig. 1): $\lambda_p = (1 - F_p(0))/(1 - p(0))$ and $1 - \lambda_p$ respectively are the overall relative probabilities of gain and loss (F_p is the cumulative distribution of p) so that $p = p^+\lambda_p + p^-(1 - \lambda_p)$, where $p^+(x) = p(x)/\lambda_p$ and $p^-(x) \equiv 0$ for $x > 0$, $p^-(x) = p(x)/(1 - \lambda_p)$ and $p^+(x) \equiv 0$ for $x < 0$, and $p^+(0) = p^-(0) = p(0)$ if $0 < \lambda_p < 1$. If, on the other hand, $\lambda_p = 1$ or $\lambda_p = 0$, then $p^{\pm} = p$, respectively. The expected gain (loss) is $\mu_p^{\pm} = \sum_{x \in S_p} x p^{\pm}(x)$, where S_p is the support of p and $\mu_p = \mu_p^+\lambda_p + \mu_p^-(1 - \lambda_p)$. Finally, for every $p \in P$ with $0 < \lambda_p < 1$, there exists a unique p^0, $p^0 = p^+\lambda_p^0 + p^-(1 - \lambda_p^0)$, so that p^0 and s are "isoneutral" in preference (i. e., $c_p^0 = c_s = 0$), where λ_p^0 is determined by

$$\frac{\mu_s}{(\mu_s^+ - \mu_s^-)\sqrt{\lambda_s(1 - \lambda_s)}} = \frac{\mu_p^+\lambda_p^0 + \mu_p^-(1 - \lambda_p^0)}{(\mu_p^+ - \mu_p^-)\sqrt{\lambda_p^0(1 - \lambda_p^0)}} \tag{2}$$

(Geiger, 2008, Sec. 5).

The basic equations of the approach involve a probability-dependent utility function $u(p, x)$ and the functional U: $P \to \mathbb{R}$ so that

$$-\frac{u(p, \mu_p^+)}{u(p, \mu_p^-)} = \frac{\varepsilon(1 - \lambda_p) + (1 - \varepsilon)\left(1 - \lambda_p^0\right)}{\varepsilon\lambda_p + (1 - \varepsilon)\lambda_p^0}, \quad \mu_p^- < 0 \tag{3}$$

$$U(p) = (1 - p(0))(u(p, \mu_p^-)(1 - \lambda_p) + u(p, \mu_p^+)\lambda_p), \tag{4}$$

$$U(p) = U(q) \Leftrightarrow u(p, x) = u(q, x), \quad p \in P, q \in P, x \in I \tag{5}$$

$$u(p, -x) = A(p)u(p, x), \quad x \geq 0, A > 0 \tag{6}$$

where $A(p)$ is a parameter-dependent elementary algebraic function (parameters ε, s; see Geiger, 2008, pp. 131–132). In Eq. (5), $\varepsilon(T_p > T_s) = \varepsilon(T_q > T_s)$ is tacitly assumed. U and u are unique up to positive affine transformations of the utility scale. They are normalised to $u(p, 0) = 0$, $u(p, -1) = -1$, $U(0) = 0$ and $U(-1) = -1$. It is important to note that in Eq. (4) the assessment of p in terms of cumulative gains and losses is a consequence of the general principles underlying the approach, notably the assumption of a neutral reference point and an axiom of *status quo* dependence of risk preferences. As for the empirical significance of the cumulative probabilities of success and failure in risky choice, see Fennema and Wakker (1997), Payne (2005) and Diecidue and van de Ven (2008).

The dependence of u on x has been made explicit elsewhere. Here, it suffices to note that $u(p, x)$ is everywhere smooth and strictly increasing in x so that c_p exists for all $p \in P$ (Geiger, 2002, 2008). For $a > 0$, let $X^a = aX$ and $p^a(X^a = ax) = p(X = x)$ so that $p^a(0) = p(0)$, $\lambda_p^a = \lambda_p$ and $\lambda_p^{0,a} = \lambda_p^0$ by construction, where "λ_p^a" and "$\lambda_p^{0,a}$" are obvious notations. Then,

$$U(p^a) = U(p), \quad u(p^a, x) = u(p, x), \quad p \in P, x \in I \tag{7}$$

in agreement with the equivalence (5). Eqs. (7) imply that utility preferences are invariant to positive homogeneous linear transformations of the x-axis. Respectively denoting the certainty equivalents of $p^{\pm}$ and p^0 by $c_p^{\pm}$ and c_p^0, one altogether has

$$\mu_p^- = c_p^- < c_p^0 = c_s = 0 < c_p^+ = \mu_p^+ \tag{8}$$

Note that $c_p^{\pm} = \mu_p^{\pm}$ does not necessarily mean risk neutrality of $p^{\pm}$ since, in the representation $p = p^+\lambda_p + p^-(1 - \lambda_p)$, $p^{\pm}$ are degenerate risks which respectively give $\mu_p^{\pm}$ with probabilities $\lambda_p = 1$ and $\lambda_p = 0$. For a degenerate risk x, one always has $c_x = \mu_x = x$ even if the associated utility function is convex or concave, that is, non-neutral.

3. The pricing of risk

Fig. 1 shows the indifference pattern in a probability triangle Δ, $\Delta \subset P$, with the vertices $p_1 = p_1^-$, $p_2 = 0$, $p_3 = p_3^+$. The indifference lines are straight and intersect in a point Z outside Δ on the extended isoneutral line connecting 0 and p^0. Z is uniquely determined, besides by p, by the exogenous parameters ε and s. For every p, $p \in \Delta$, with $0 < \lambda_p < 1$, there exists a unique q, $q \in \Delta$, so that $q = q^{\pm}$ and $U(p) = U(q^{\pm})$ if $\lambda_p \gtrless \lambda_p^0$. In either case,

$$\frac{\mu_p^+}{\mu_p^-} = \frac{\mu_q^+}{\mu_q^-} \tag{9}$$

If $\lambda_p = \lambda_p^0$, then $q = 0$. Considering (8), one also has

$$c_p = c_q = c_q^{\pm} = \mu_q^{\pm} = \frac{(1 - q(0))\mu_p^{\pm}}{(1 - p(0))} \tag{10}$$

Letting $a = -1/\mu_p^-$, replacing $\mu_p^{\pm}$ by $a\mu_p^{\pm}$ in Eq. (4) and considering Eqs. (7) gives

$$U(p^a) = U(p) = (1 - p(0))u(p, -\mu_p^+/\mu_p^-)\left(\frac{-1 + \lambda_p}{u(p, -\mu_p^+/\mu_p^-)} + \lambda_p\right) \tag{11}$$

where $u(p, -1) = -1$ has been used. Assume $q = q^+$ so that $\lambda_q = 1$, and let $\lambda_p > \lambda_p^0$. It follows

$$\begin{aligned} U(p) &= U(q^+) = (1 - q(0))u(q, \mu_q^+) \quad \text{by Eq. (4)} \\ &= (1 - q(0))u(q^a, -\mu_q^+/\mu_q^-), a = -1/\mu_q^- \quad \text{by Eqs. (7), (11)} \\ &= (1 - q(0))u(p^a, -\mu_p^+/\mu_p^-) \quad \text{by Eqs. (5), (9)} \\ &= (1 - q(0))u(p, -\mu_p^+/\mu_p^-) \quad \text{by Eq. (7)} \\ &= (1 - p(0))u(p, -\mu_p^+/\mu_p^-)c_p/\mu_p^+ \quad \text{by Eq. (10)} \end{aligned} \tag{12}$$

[1] The interpretation of ε as $\varepsilon(T_p > T_s)$ is as in Geiger (2012), but differs from that in Geiger (2002, 2008), where ε is defined as the complementary probability $\varepsilon(T_p \leq T_s)$ obtained by substituting $\varepsilon \to 1 - \varepsilon$. Our formal results remain unaffected by this change in denotation, but the present specification of ε is appropriate in applications as it gives the correct limiting behaviour of the utility function in the boundary cases $\varepsilon = 0$ and $\varepsilon = 1$ (Geiger 2012).