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Continuous Optimization

Algebraic simplex initialization combined with the nonfeasible basis method

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ABSTRACT

We propose, in this paper, a new method to initialize the simplex algorithm. This approach does not involve any artificial variables. It can detect also the redundant constraints or infeasibility, if any. Generally, the basis found by this approach is not feasible. To achieve feasibility, this algorithm appeals to the nonfeasible basis method (NFB). Furthermore, we propose a new pivoting rule for NFB method, which shows to be beneficial in both numerical and time complexity. When solving a linear program, we develop an efficient criterion to decide in advance which algorithm between NFB and formal nonfeasible basis method seems to be more rapid. Comparative analysis is carried out with a set of standard test problems from *Netlib*. Our computational results indicate that the proposed algorithm is more advantageous than two-phase and perturbation algorithm in terms of number of iterations, number of involved variables, and also computational time.

1. Introduction

The simplex algorithm, proposed by Dantzig (1951), is the most used method to solve linear programming problems. A feasible basis is a principal data for starting the latter. Outside the canonical form, it is not easy in practice to exhibit an initial basis even infeasible. Many methods exist in the literature for initializing the simplex algorithm. Two-phase and big M are the most known methods to find an initial feasible basis. They require both artificial variables to get the identity matrix as initial basis. But the addition of artificial variables leads to increase the size of the problem. Stojkovic and Stanimirovic (2001) used “the cosine criterion” to get an initial basis for the simplex algorithm. Paparrizos, Samars, and Stephanides (2003) developed a method that starts resolution by an infeasible basis. This method involves two artificial variables and two big numbers. Csizmadia, Illés, and Nagy (2012) developed the concept of s-monotone index selection rule which unifies the finiteness proof of some anti-cycling pivot rules. The work by Hu (2007) gives a new technique for searching a basis, not necessary feasible, based on LU decomposition. In order to reach feasibility, the author used the perturbation method (Pan, 2000). The main idea of this technique consists in perturbing the economical function; so that the vector of reduced costs becomes nonpositive. This approach applies thenceforward the dual-simplex algorithm. The optimal basis found by the perturbed problem is feasible but not necessarily optimal for the original problem. At this stage, the standard simplex algorithm can be executed normally. Recently, Nabli (2009) suggested a method, termed nonfeasible method (NFB), in order to construct an initial feasible solution from an infeasible one. This method operates without artificial variables or a big M number and without any perturbation in the objective function. The outcome of the feasibility is via a modification of the structure of the simplex algorithm in the choice of the entering and leaving variables. This method is a new approach which is completely different from the standard simplex method and also from the dual-simplex algorithm. In the same paper (Nabli, 2009), Nabli introduced the notion of formal tableau. As a consequence, he developed another new method called formal nonfeasible basis and denoted by FNFB. Nabli, Chahdoura, and Dammak (2013), have combined the method of Hu (2007) with the nonfeasible basis method.

Extracting a basis even infeasible is not a simple issue. Also, to find a basic solution of the linear system $My = b$, related to a standard form, we need to eliminate first all redundant constraints, if any. Otherwise it is not possible to find a basis of order m, where m is the number of rows in matrix M. In this paper, we introduce a new approach which initializes the simplex algorithm by a feasible basis of adequate order. We modify slightly the nonfeasible basis method. When solving a linear program, we develop an efficient criterion to decide in advance which algorithm between NFB and FNFB seems to be less consuming in number of iterations.

The paper is organized as follows. After this introduction, the simplex algorithm is briefly described. In Section 3, we explain our

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algorithm which is able to extract a basis from the matrix governing the constraints after detecting possible redundancy. An example is proposed to illustrate our approach. In Section 4, the nonfeasible basis method is recalled and a slight modification on its pivoting rule is carried out. We describe likewise our choice criterion. Section 5 is dedicated to a comparative study between our approach, two-phase method and the method of Hu (2007). The comparison of different methods is based on the number of involved iterations and the time complexity. Finally, Section 6 summarizes our contribution.

2. Simplex algorithm

It is well known that the executing of simplex algorithm is restricted only on standard forms. Each linear program, in canonical or general form, must be beforehand written in standard form to ensure its implementation by the simplex algorithm. Any linear program in standard form is expressed as follows:

$$\begin{cases} \max(or\min)[Z(y) = c^*y] \\ My = b \\ y \geq 0_{\mathbb{R}^n}, \end{cases}$$

where M is a $(m,\ n)$ matrix satisfying $1 \leq m < n$, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ are assumed to be column vectors. The mathematical symbol * stands for the transpose operator. The following notations will be used throughout this paper: $M_{.j} = M(:, j)$ the jth column of M; $M_{i.} = M(i, :)$ the ith row of M; M_{ij} the (i, j) entry of M; $M(i{:}\ j, :)$ all rows from i to j of the matrix M and $M(:, i{:}\ j)$ all columns from i to j of the matrix M.

We assume that the rank of matrix M is maximal equal to m, $(\text{rank}(M) = m)$, otherwise all linearly dependent row vectors must be imperatively removed. This hypothesis ensures the existence of a nonsingular sub-matrix of dimension m called basis, which is commonly denoted by B. The matrix M is partitioned as $M = [B\ N]$, where N is the matrix composed of the remaining columns. Let J_B be the set of basic variable indices and $J_N = \{1, \ldots, n\}\backslash J_B$ the set of nonbasic variable indices. According to the partition J_B and J_N, the data of linear problem can be expressed as $c^* = (c_B^*\ \ c_N^*)$ and $y = \binom{y_B}{y_N}$. The vector $y_B = (y_i, i \in J_B)$ is composed by the basic variables associated to the basis B and $y_N = (y_i,\ i \in J_N)$ by the nonbasic variables. A solution $y = \binom{y_B}{y_N}$ is feasible if and only if it satisfies the constraints $My = b$ and $y \geq 0_{\mathbb{R}^n}$, otherwise it is called infeasible. Being given a basis B, the associated solution $y = \binom{y_B}{y_N} = \binom{B^{-1}b}{0_N}$ is called a basic solution, it is feasible if and only if $B^{-1}b \geq 0_B$. Its objective function value is equal to $Z\binom{B^{-1}b}{0_N} = c_B^* B^{-1} b$. The reduced cost vector associated to the basis B is defined as follows:

$$w_N^* = \begin{cases} c_N^* - c_B^* B^{-1} N, & \text{if “max”} \\ -c_N^* + c_B^* B^{-1} N, & \text{if “min”} \end{cases}$$

It is well-known that, under the hypothesis of feasibility of B, the condition $w_N^* \leq 0_N^*$ is sufficient to state that B is an optimal basis or equivalently $\binom{B^{-1}b}{0_N}$ is an optimal solution. It becomes necessary in case of non-degeneracy $(B^{-1}b > 0_B)$. If the optimality condition is not satisfied, $(w_N^* \nleq 0_N^*)$, an adjacent feasible basic solution $\binom{B'^{-1}b}{0_{N'}}$ admitting a better objective value (Nabli, 2006; Wolfe, 1985), is selected. The change from B to B' is done by swapping a column of B with one of N. So there are two operations for this exchange. The first is to determine the index column of N which must enter in J_B. This index is chosen among the set $\{j \in J_N / w_j = \pm(c_j - c_B^* B^{-1} N_{.j}) > 0\}$. Generally, this set is not reduced to a singleton so there are several choices. There are many rules (Dantzig, 1963; Dantzig & Thapa, 1997; 2003), called pivoting rule, to fix the entering variable. Among the rules we choose steepest-edge rule, proposed by Goldfarb and Ried (1977). It consists in choosing the maximum reduced cost vector normalized; and determining the index $s \in \{1, \ldots, n-m\}$ for which this maximum is reached:

$$s = \text{argmax}\left\{\frac{w_j}{\|\eta_j\|} / w_j > 0, \text{for } j = 1, \ldots, n-m\right\}, \text{ where}$$

$$\eta_j = \begin{pmatrix} -B^{-1}Ne_j \\ e_j \end{pmatrix}.$$

In the identity above, e_j designates the jth vector of the canonical basis of $\mathbb{R}^{n-m}$. The steepest-edge rule is adopted because it is the best pivoting rule in practice (Todd, 2002). The second operation consists in selecting one column of B, to be released from the basis, so that the new basic solution remains feasible. This purpose is achieved by considering the leaving variable index:

$$r = \text{argmin}\left\{\frac{(B^{-1}b)_i}{(B^{-1}N_{.s})_i} / (B^{-1}N_{.s})_i > 0, \text{for } i = 1, \ldots, m\right\}.$$

After permuting the sth column of N with the rth column of B, the data Z, w_N^*, $B^{-1}b$ and $B^{-1}N$ are updated. This procedure is repeated until reaching the optimality condition $w_N^* \leq 0_N^*$.

For easier handling practice, the elements involved in the simplex algorithm can be recapitulated in a table, called simplex tableau. To each basis B corresponds a simplex tableau. Here, we use the condensed form (Wolfe, 1985), which is a reduced size tableau of dimension $(m+1) \times (n-m+1)$ whereas the standard simplex tableau is of dimension $(m+1) \times (n+1)$:

$$\begin{array}{cc|c} y_{j_1}\ \cdots\ y_{j_{n-m}} & & \\ \begin{array}{|c|c|}\hline B^{-1}N & B^{-1}b \\ \hline w_N^* & \mp Z \\ \hline \end{array} & & \begin{array}{c} y_{i_1} \\ \vdots \\ y_{i_m} \end{array} \end{array}$$

The variables $y_{i_1}, \ldots, y_{i_m}$ correspond to the basic variables associated to the current basis B and $y_{j_1}, \ldots, y_{j_{n-m}}$ are the nonbasic variables. The interior elements constituting this tableau are concatenated in one matrix denoted by H:

$$H = \begin{pmatrix} B^{-1}N & B^{-1}b \\ w_N^* & \mp Z \end{pmatrix}.$$

If the pivot is H_{rs} for some iteration, then the entries composing the new matrix for the subsequent iteration satisfy the following expression:

$$H_{is} \leftarrow \begin{cases} -\frac{H_{is}}{H_{rs}} & \text{for } i \neq r \\ \frac{1}{H_{rs}} & \text{for } i = r \end{cases} \quad \begin{cases} H_{rj} \leftarrow \frac{H_{rj}}{H_{rs}}, & j \neq s \\ H_{ij} \leftarrow H_{ij} - \frac{H_{is}H_{rj}}{H_{rs}}, & j \neq s\ \&\ i \neq r \end{cases} \tag{1}$$

When solving a linear program by the simplex method, we are not forced to calculate all the entries of the simplex tableau, only involved elements are computed. The revised simplex algorithm consists of avoiding the wasting time on calculating the elements which are not used.

3. Initial basis

As mentioned before, to apply the simplex algorithm, a nonsingular matrix B of dimension m is required. But in general, it is difficult to detect at first sight such a basis. For a canonical form, the constraints are written in the form $Ax \leq b$ and $x \geq 0_{\mathbb{R}^p}$, so the matrix M associated to the standard form is none other than $M = [A\ \ I]$, where I is the identity matrix. The size of the matrix I is described by the context. Therefore, if $b \geq 0_{\mathbb{R}^m}$, the matrix $B = I$ is a feasible basis since $B^{-1}b = b \geq 0_{\mathbb{R}^m}$. In other words, the slack variables are basic ones for a canonical form satisfying $b \geq 0_{\mathbb{R}^m}$, otherwise $(b \ngeq 0_{\mathbb{R}^m})$ or in the case where the linear program is not written in canonical form, it is often difficult to extract a feasible basis. For some problems, the rank of the linear system $My = b$ is lower than m: $\text{rank}(M) < m$. In this case, either