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Decision Support

Mean-variance model for portfolio optimization problem in the simultaneous presence of random and uncertain returns

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ABSTRACT

The determination of security returns will be associated with the validity of the corresponding portfolio selection models. The complexity of real financial market inevitably leads to diversity of types of security returns. For example, they are considered as random variables when available data are enough, or they are considered as uncertain variables when lack of data. This paper is devoted to solving such a hybrid portfolio selection problem in the simultaneous presence of random and uncertain returns. The variances of portfolio returns are first given and proved based on uncertainty theory. Then the corresponding mean-variance models are introduced and the analytical solutions are obtained in the case with no more than two newly listed securities. In the general case, the proposed models can be effectively solved by Matlab and a numerical experiment is illustrated.

1. Introduction

Portfolio selection problem is to consider how to allocate the fund among the candidate set of risky securities to maximize return and to minimize risk. Since it was originated by Markowitz (1952), there has been a large number of literature to be devoted to developing the problem. Same as the framework of Markowitz's mean-variance model, most of the works are undertaken to measure the investment return and risk by expected value and variance of return of an portfolio, respectively. In spite of the computational difficulty, the model is still successfully applied in reality and stimulate the development of finance.

In portfolio theory, the security returns are considered as random variables and their characteristics such as expected value and variance are calculated based on the sample of available historical data. It remains valid when there are plenty of data such as in the developed financial market. However, there may be lack of enough transaction data in some emerging markets. In the situation, some researchers regarded security returns as fuzzy variables estimated by experienced experts, and developed fuzzy portfolio optimization theory. More specifically, fuzzy portfolio optimization has been studied based on three different methods: fuzzy set theory (Gupta, Mehlawat, & Saxena, 2008), possibility theory (Carlsson, Fullér, & Majlender, 2002; Zhang, Wang, Chen, & Nie, 2007) and credibility theory (Huang, 2006; Qin, Li, & Ji, 2009).

Although fuzzy portfolio optimization provided alternatives to estimate security returns when lack of data, fuzzy theory suffers from criticism since a paradox will appear when fuzzy variable is used to describe the security returns (see Huang & Ying, 2013). In order to better describe the subjective imprecise quantity, Liu (2007) established uncertainty theory as another alternative way to estimate indeterministic quantities subject to experts' estimations. Based on this framework, much works are undertaken to develop the theory (Chen & Dai, 2011; Gao, 2009; Yao, 2012; You, 2009) and related practical applications (Li, Peng, & Zhang, 2013; Liu, 2010b; 2013; Qin & Kar, 2013). In particular, uncertainty theory is also applied to model portfolio selection. Qin, Kar, and Li (2015) first considers mean-variance model in uncertain environment. After that, Huang (2012) established a risk index model for uncertain portfolio selection, and Huang and Ying (2013) further employed the criterion to consider portfolio adjusting problem. Different from risk index model, Liu and Qin (2012) presented semiabsolute deviation of uncertain variable to measure risk and formulated the corresponding return-risk models, and Qin, Kar, and Zheng (In Press) employed the concept to describe and model portfolio adjusting problem. In addition to these single-period optimization models, Huang and Qiao (2012) also modeled the multi-period problem, and Zhu (2010) founded the uncertain optimal control and applied it to solve a continuous-time uncertain portfolio selection problem.

Whether the classical portfolio selection or fuzzy/uncertain one, security returns are considered as the same type of variables. In other words, security returns are assumed to be either random variables or fuzzy/uncertain variables. As stated above, the former makes use of the historical data and the latter makes use of the experiences of

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experts. However, the actual situation is that the securities having been listed for a long time have yielded a great deal of transaction data. For these "existing" securities, statistical methods are employed to estimate their returns, which implies that it is reasonable to assume that security returns are random variables. For some newly listed securities, there are lack of data or there are only insufficient data which cannot be used to effectively estimate the returns. Therefore, the returns of these newly listed securities need to be estimated by experts and thus are considered as uncertain variables. If an investor faces as such complex situation with simultaneous appearance of random and uncertain returns, how should he/she select a desirable portfolio to achieve some objectives? This paper attempts to consider the hybrid portfolio selection problem and establish mathematical models by means of uncertain random variable which was proposed by Liu (2013a) for modeling complex systems with not only uncertainty but also randomness. Based on this spirit, uncertain random programming is studied by Liu (2013b) and extended by Zhou, Yang, and Wang (2014), also applied to graph and network (Liu, 2014), risk analysis (Liu & Ralescu, 2014) and so on.

The rest of the paper is organized as follows. In Section 2, we review the necessary preliminaries related to uncertain measure, uncertain variable and uncertain random variable. Section 3 in detail describes the problem and notations and then gives two assumptions which are commonly used in portfolio analysis. Further, the formulas of variances of portfolio returns are given and proved. Section 4 considers the formulations of mean-variance models and discusses the solution procedures. A numerical experiment is illustrated to the application of the proposed models in Section 5. Finally, some conclusions are given in Section 6.

2. Preliminaries

In this section, uncertain measure, uncertain variable and uncertain random variable are introduced for the completeness of the paper. More details about existing measures of uncertainties the readers may consult (Liu, 2012). In 2007, Liu (2007) proposed the concept of uncertain measure to indicate the belief degree that a possible event happens. Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ.

Definition 1 (Liu 2007). A set function $\mathcal{M}: \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies: (1) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ; (2) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event $\Lambda \in \mathcal{L}$; (3) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have $\mathcal{M}\{\bigcup_{i=1}^{\infty} \Lambda_i\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$. The triple $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertain space.

In order to provide the operational law, Liu (2010a) further provided the product axiom as follows. Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively. Next, the concept of uncertain variable was defined to describe a quantity with uncertainty.

Definition 2 (Liu, 2007). An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\} \in \mathcal{L}.$$

Definition 3 (Liu, 2007, 2010a). For any $x \in \Re$, the uncertainty distribution of an uncertain variable ξ is defined by $\Phi(x) = \mathcal{M}\{\xi \le x\}$. It is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and

$$\lim_{x\to-\infty} \Phi(x) = 0, \quad \lim_{x\to+\infty} \Phi(x) = 1.$$

The inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ if it exists and is unique for each $\alpha \in (0, 1)$.

Inverse uncertainty distribution plays a crucial role in the operations of independent uncertain variables. Next we introduce some commonly used uncertainty distributions. An uncertain variable is called linear if it has a linear uncertainty distribution

$$\Phi(r) = \begin{cases} 0, & \text{if } r \le a, \\ (r-a)/(b-a), & \text{if } a \le r \le b, \\ 1, & \text{if } r \ge b, \end{cases}$$

denoted by $\mathcal{L}(a, b)$ where a and b are real numbers with $a < b$. An uncertain variable is called zigzag if it has a zigzag uncertainty distribution

$$\Phi(r) = \begin{cases} 0, & \text{if } r \le a, \\ (r-a)/2(b-a), & \text{if } a \le r \le b, \\ (x+c-2b)/2(c-b), & \text{if } b \le r \le c, \\ 1, & \text{if } r \ge c, \end{cases}$$

denoted by $\mathcal{Z}(a, b, c)$ where a, b, c are real numbers with $a < b < c$. An uncertain variable is called normal if it has a normal uncertainty distribution

$$\Phi(r) = \left(1 + \exp\left(\frac{\pi(e-r)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad r \in \Re,$$

denoted by $\mathcal{N}(e, \sigma)$ where e and σ are real numbers with $\sigma > 0$. These three uncertainty distributions are all regular.

Lemma 1 (Liu, 2010a). *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_n$, then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable with an uncertainty distribution*

$$\Psi(\alpha) = \sup_{f(x_1, x_2, \ldots, x_n)} \min_{1\le i\le n} \Phi_i(x_i),$$

and with an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)\right).$$

Definition 4 (Liu, 2007). The expected value of an uncertain variable ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge x\}\mathrm{d}x - \int_{-\infty}^{0} \mathcal{M}\{\xi \le x\}\,\mathrm{d}x$$

provided that at least one of the two integrals exists.

Lemma 2 (Liu, 2010a). *Let ξ be an uncertain variable with regular uncertainty distribution Φ. If the expected value $E[\xi]$ exists, then $E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\,\mathrm{d}\alpha$.*

Uncertain random variable is employed to describe a complex system with not only uncertainty but also randomness. Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space, and $(\Omega, \mathcal{P}, \Pr)$ a probability space. The product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{P}, \Pr)$ is called a chance space. We have the following definition.

Definition 5 (Liu, 2013a). Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{P}, \Pr)$ be a chance space, and let $\Theta \in \mathcal{L} \times \mathcal{P}$. Then the chance measure of uncertain random event Θ is defined as

$$\mathrm{Ch}\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \ge r\}\,\mathrm{d}r.$$

Definition 6 (Liu, 2013a). An uncertain random variable is a function ξ from the chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{P}, \Pr)$ to the set of real numbers, i.e., $\xi \in B$ is an event in $\mathcal{L} \times \mathcal{P}$ for any Borel set B. Its chance distribution is defined by $\Phi(x) = \mathrm{Ch}\{\xi \le x\}$ for any $x \in \Re$.